

Introduction to Real Analysis – Homework 7

Borel sets and measurable functions

Write $\mathcal{B}(Y)$ for the σ -algebra generated by the open subsets of Y . Real-valued functions have target $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

1. **From level sets to Borel sets.** Let $(X, \mathcal{A})$ be a measurable space and $f : X \rightarrow \mathbb{R}$ a function.

(a) Prove that

$$\mathcal{B}(\mathbb{R}) = \sigma(\{(a, \infty) : a \in \mathbb{R}\}).$$

Hint. Obtain $[b, \infty)$ by a countable intersection, then obtain open intervals. Every open subset of $\mathbb{R}$ is a countable union of intervals with rational endpoints.

(b) Show that $\mathcal{D} = \{B \subset \mathbb{R} : f^{-1}(B) \in \mathcal{A}\}$ is a σ -algebra on $\mathbb{R}$.

(c) Deduce that f is measurable if and only if $\{x \in X : f(x) > a\} \in \mathcal{A}$ for every $a \in \mathbb{R}$. Explain why, for $X = \mathbb{R}^d$ and $\mathcal{A}$ the Lebesgue σ -algebra, this is exactly our earlier definition of a measurable function.

(d) Prove the analogous criterion for $f : X \rightarrow Y = [0, \infty]$, using the rays $\{y \in Y : y > a\}$, $a \in \mathbb{R}$. Also show that $\{x : f(x) = \infty\}$ is measurable whenever f is.

Hint. A countable basis for the usual topology on Y consists of $[0, b)$, (a, b) and $(a, \infty]$ with rational endpoints $0 \leq a < b$. Treat the endpoints 0 and ∞ explicitly.

2. **A finite measurable space.** Let $X = \{1, 2, 3, 4, 5\}$ and

$$\mathcal{A} = \sigma(\{\{1, 2, 3\}, \{3, 4, 5\}\}).$$

Describe the sets in $\mathcal{A}$, and characterize all measurable functions $f : X \rightarrow \mathbb{R}$.

3. **Completeness and null changes.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that the following are equivalent:

(a) The measure space is complete.

(b) Whenever $f : X \rightarrow \mathbb{R}$ is measurable and $g : X \rightarrow \mathbb{R}$ agrees with f outside a measurable null set, g is measurable.

Hint. For the reverse implication, compare 0 and $\mathbf{1}_E$ for an arbitrary subset E of a measurable null set.