

Quiz 9: Solutions

Give exact answers and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. Use the material covered through Section 5 of Lecture 10. For local extrema, consider only interior points of the stated domain.

1. Find real numbers a, b such that

$$f(x) = x^3 + ax^2 + bx$$

satisfies $f(1) = 2$ and has a local maximum at 1. Use Fermat's theorem to find the possible coefficients, then verify the local maximum directly by examining $f(1+h) - f(1)$ for small nonzero h .

Solution.

The value condition gives $1 + a + b = 2$, or $a + b = 1$. Since a polynomial is differentiable, Fermat's theorem gives the necessary condition

$$f'(1) = 3 + 2a + b = 0.$$

Solving these two equations yields $\boxed{a = -4, b = 5}$. For this choice,

$$f(1+h) - f(1) = h^3 - h^2 = h^2(h-1).$$

If $0 < |h| < 1/2$, then $h^2 > 0$ and $h-1 < 0$. Therefore $f(1+h) < f(1) = 2$, proving that 1 is a strict local maximum. The coefficients satisfy both requirements.

2. For $x \neq 0$, let

$$H(x) = \arctan x + \arctan(1/x).$$

Show that $H'(x) = 0$ throughout its domain. Determine $H(x)$ on each of the intervals $(-\infty, 0)$ and $(0, \infty)$, and explain why the constants need not agree.

Solution.

For $x \neq 0$, the chain rule gives

$$H'(x) = \frac{1}{1+x^2} + \frac{-1/x^2}{1+1/x^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0.$$

The function is differentiable on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. The zero-derivative criterion therefore makes it constant on each interval. Evaluate at one point in each:

$$H(-1) = 2 \arctan(-1) = -\frac{\pi}{2}, \quad H(1) = 2 \arctan 1 = \frac{\pi}{2}.$$

Consequently

$$H(x) = \begin{cases} -\pi/2, & x < 0, \\ \pi/2, & x > 0. \end{cases}$$

The domain $\mathbb{R} \setminus \{0\}$ is not an interval. The zero-derivative criterion applies separately to the two intervals and does not require their constants to agree.