

## Quiz 8: Solutions

Give exact answers unless an approximation is requested, and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. Use the results covered through Section 8 of Lecture 9. For local extrema, consider only interior points of the stated domain. Use the convention  $\operatorname{arcsec} x = \arccos(1/x)$ , with values in  $[0, \pi] \setminus \{\pi/2\}$ .

1. The linearization of a differentiable function  $f$  at  $a = 2$  is

$$L_f(x) = 5 - 3(x - 2).$$

Find the linearization of  $g(x) = \ln f(x)$  at  $a = 2$ . Justify that  $g$  is defined near 2.

**Solution.**

By the definition of linearization,  $f(2) = 5$  and  $f'(2) = -3$ . Differentiability implies continuity at 2. Consequently, there is a  $\delta > 0$  such that  $|x - 2| < \delta$  implies  $|f(x) - 5| < 5/2$ . In particular,  $f(x) > 5/2 > 0$  there, so  $g(x) = \ln f(x)$  is defined near 2.

The chain rule gives

$$g(2) = \ln 5, \quad g'(2) = \frac{f'(2)}{f(2)} = -\frac{3}{5}.$$

Thus the linearization is

$$L_g(x) = \ln 5 - \frac{3}{5}(x - 2).$$

2. Let  $f$  be continuous on  $[a, b]$ , where  $a < b$ , and suppose  $f(x) \neq 0$  for every  $x \in [a, b]$ .
- (a) Prove that there is a number  $m > 0$  such that  $|f(x)| \geq m$  for every  $x \in [a, b]$ , and deduce that  $1/f$  is bounded.
- (b) Does the same conclusion hold for every continuous nonzero function on  $(0, 1)$ ? Prove it or give a counterexample.

*Hint for (a):* apply the Extreme Value Theorem to  $|f|$ .

**Solution.**

(a) The function  $|f|$  is continuous on the closed, bounded interval  $[a, b]$ . By the Extreme Value Theorem, it attains its minimum at some  $c \in [a, b]$ . Set  $m = |f(c)|$ . Since  $f(c) \neq 0$ , we have  $m > 0$ , and

$$|f(x)| \geq m \quad \text{for every } x \in [a, b].$$

Taking reciprocals of these positive quantities gives

$$\left| \frac{1}{f(x)} \right| \leq \frac{1}{m} \quad \text{for every } x \in [a, b].$$

Hence  $1/f$  is bounded.

(b) No. Let  $f(x) = x$  on  $(0, 1)$ . This function is continuous and never zero. For any  $m > 0$ , choose  $x = \min\{m/2, 1/2\}$ . Then  $x \in (0, 1)$  and  $0 < f(x) < m$ , so there is no positive lower bound for  $|f|$ . Moreover, for any  $K > 0$ , the point  $x = 1/(K+1)$  lies in  $(0, 1)$  and satisfies

$$\frac{1}{f(x)} = K+1 > K.$$

Thus  $1/f$  is unbounded on  $(0, 1)$ .