

Quiz 7: Solutions

Give exact answers and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. You may use the results covered through Lecture 8, including the chain rule, the inverse-function derivative theorem, and the derivatives of $\ln$ and $\arcsin$. For implicit curves, you may assume that the local branches used are differentiable wherever the differentiated equation has a nonzero coefficient of the derivative being solved for.

1. For the curve

$$e^y + xy = e^x,$$

find dy/dx where it can be obtained by solving the differentiated equation. Check that $(0, 0)$ lies on the curve and find its tangent line there.

Solution.

Differentiating along a branch gives

$$e^y y' + y + xy' = e^x, \quad (e^y + x)y' = e^x - y.$$

Therefore, wherever $e^y + x \neq 0$,

$$\boxed{\frac{dy}{dx} = \frac{e^x - y}{e^y + x}}.$$

At $(0, 0)$, the original equation reads $1 = 1$, so the point is on the curve. The coefficient $e^y + x$ equals 1, and $y' = 1$ there. The tangent line is $\boxed{y = x}$.

2. The function

$$\tan : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$$

is one-to-one and onto, with inverse $\arctan$. Use the inverse-function derivative theorem to find $\frac{d}{dx} \arctan x$, expressing your answer entirely in terms of x . Explain why the derivative exists for every real x .

Solution.

For any $x \in \mathbb{R}$, let $y = \arctan x$. Then $y \in (-\pi/2, \pi/2)$ and $\tan y = x$. On that interval, tangent is continuous and strictly increasing, and its derivative is $\sec^2 y > 0$. The inverse-function derivative theorem therefore gives

$$\frac{d}{dx} \arctan x = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1 + \tan^2(\arctan x)} = \boxed{\frac{1}{1 + x^2}}.$$

Every real x has an inverse value in this open interval, where the tangent derivative is finite and nonzero. Thus the inverse derivative exists for every real x ; its displayed denominator $1 + x^2$ is always positive.