

1. Antiderivatives

Definition. Let f be defined on an open interval I . An *antiderivative* of f on I is a differentiable function F on I such that

$$F'(x) = f(x) \quad (x \in I).$$

For example, x^3 and $x^3 + 7$ are antiderivatives of $3x^2$. Every constant added to an antiderivative disappears upon differentiation.

Theorem. If F is one antiderivative of f on an interval I , then all antiderivatives of f on I have the form

$$F(x) + C, \quad C \in \mathbb{R}.$$

Proof. Every $F + C$ has derivative f . Conversely, if $G' = f$, put $H = G - F$. Then $H' = 0$. Given $u < v$ in I , the entire interval $[u, v]$ lies in I , and H is continuous on $[u, v]$ and differentiable on (u, v) . The Mean Value Theorem gives

$$H(v) - H(u) = H'(c)(v - u) = 0$$

for some $c \in (u, v)$. Thus H is constant on I . $\square$

Example

All antiderivatives of $3x^2 - 4x + 2$ on $\mathbb{R}$ are

$$F(x) = x^3 - 2x^2 + 2x + C.$$

Differentiation verifies that these functions work; the theorem shows that there are no others.

The interval hypothesis. On either half-line, $(\ln |x|)' = 1/x$. On $\mathbb{R} \setminus \{0\}$, all solutions of $F'(x) = 1/x$ have the form

$$F(x) = \begin{cases} \ln(-x) + C_-, & x < 0, \\ \ln x + C_+, & x > 0, \end{cases}$$

where C_- and C_+ are independent. The MVT cannot be applied across the missing point 0.

A condition $F(x_0) = y_0$ determines the constant on an interval once one antiderivative is known. It does not determine independent constants on other components of a disconnected domain.

2. Computing antiderivatives

Each entry follows by differentiating the middle column. Work on an interval contained in the indicated domain; add a constant for the general antiderivative.

$f(x)$	One antiderivative	Domain or restriction
x^r	$x^{r+1}/(r+1)$	$x > 0, r \neq -1$
$1/x$	$\ln x $	$x \neq 0$
e^x	e^x	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\sec^2 x$	$\tan x$	$\cos x \neq 0$
$1/(1+x^2)$	$\arctan x$	$\mathbb{R}$
$1/\sqrt{1-x^2}$	$\arcsin x$	$-1 < x < 1$

For nonnegative integer powers the formula holds on all of $\mathbb{R}$. For negative integer powers other than -1 , it holds on either half-line. The positive-domain statement covers all real exponents. Linearity follows from

$$(\alpha F + \beta G)' = \alpha f + \beta g \quad \text{when } F' = f, G' = g.$$

Example

The chain rule gives the following derivative identities:

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{2} e^{2x} \right) &= e^{2x}, & \frac{d}{dx} \left(-\frac{1}{3} \cos(3x) \right) &= \sin(3x), \\ \frac{d}{dx} \ln(1+x^2) &= \frac{2x}{1+x^2}. \end{aligned}$$

Thus all antiderivatives on $\mathbb{R}$ of these three functions are, respectively,

$$\frac{1}{2} e^{2x} + C, \quad -\frac{1}{3} \cos(3x) + C, \quad \ln(1+x^2) + C.$$

For the last expression, $1+x^2 > 0$ for every real x . Each answer can be checked by differentiation.

Example

Find F on $(0, \infty)$ such that

$$F'(x) = 3x^2 - \frac{2}{x}, \quad F(1) = 4.$$

The general antiderivative is $F(x) = x^3 - 2 \ln x + C$. Since $F(1) = 1 + C = 4$,

$$\boxed{F(x) = x^3 - 2 \ln x + 3 \quad (x > 0).}$$

Differentiating and substituting $x = 1$ check both requirements. The constant theorem gives uniqueness on this interval.

3. Change in position and area

Change in position. Let $x(t)$ be the position of a moving object at time t , and let $v(t) = x'(t)$ be its velocity. Suppose v is continuous on $[0, T]$, where $T > 0$. We want to recover the change in position

$$x(T) - x(0)$$

from the velocity function.

Divide the time interval into smaller intervals:

$$0 = t_0 < t_1 < \cdots < t_n = T, \quad \Delta t_i = t_i - t_{i-1}.$$

Over a short interval, approximate the velocity by its value at a sample point $\tau_i \in [t_{i-1}, t_i]$. The corresponding approximation for the change in position is

$$x(t_i) - x(t_{i-1}) \approx v(\tau_i) \Delta t_i.$$

For constant velocity this is exact. For varying velocity, it treats the velocity as approximately constant on a short time interval.

Adding the changes in position gives an exact cancellation of the intermediate positions:

$$\begin{aligned} x(T) - x(0) &= \sum_{i=1}^n (x(t_i) - x(t_{i-1})) \\ &\approx \sum_{i=1}^n v(\tau_i) \Delta t_i. \end{aligned}$$

This suggests recovering the total change in position by taking a limit of such sums as all the time intervals become short. Negative velocities contribute negative changes in position. The quantity being approximated is displacement, not total distance travelled.

Area under a graph. Let f be a continuous nonnegative function on $[a, b]$. To approximate the area between its graph and the horizontal axis, divide $[a, b]$ into smaller intervals,

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Choose $\xi_i \in [x_{i-1}, x_i]$. A rectangle with width $\Delta x_i = x_i - x_{i-1}$ and height $f(\xi_i)$ has area $f(\xi_i) \Delta x_i$, so the total area is approximated by

$$\sum_{i=1}^n f(\xi_i) \Delta x_i.$$

We seek a limit that does not depend on the chosen subdivisions or sample points, provided every width becomes small.

Both problems lead to sums of the same form: a function value multiplied by an interval length, added over the intervals. This leads to the following definitions.

4. Partitions and Riemann sums

Definitions. A *partition* of $[a, b]$, where $a < b$, is a finite list

$$P : a = x_0 < x_1 < \cdots < x_n = b.$$

Put $\Delta x_i = x_i - x_{i-1}$. The *mesh* is the largest width:

$$\|P\| = \max_{1 \leq i \leq n} \Delta x_i.$$

Choose a sample point $\xi_i \in [x_{i-1}, x_i]$ in each subinterval. The corresponding *Riemann sum* is

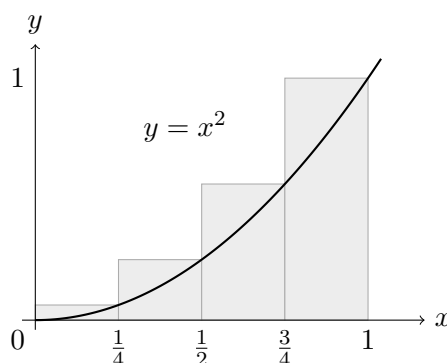
$$S(f; P, \xi) = \sum_{i=1}^n f(\xi_i) \Delta x_i.$$

The summation symbol means to add one term for each $i = 1, \dots, n$. The widths add to $b - a$.

Example

For $f(x) = x^2$ on $[0, 1]$, take four equal subintervals and their right endpoints. The sample points are $1/4, 1/2, 3/4, 1$, and

$$S = \frac{1}{4} \left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1 \right] = \frac{15}{32}.$$



For an equal partition of $[a, b]$, $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$. Right-endpoint sums are

$$\sum_{i=1}^n f(a + i\Delta x) \Delta x.$$

Mesh versus number of intervals. Many subintervals do not by themselves make every subinterval small. One could leave $[0, 1/2]$ intact and subdivide only $[1/2, 1]$. The number of intervals grows, but the mesh remains $1/2$.

5. Riemann integrability

Let f be a bounded function on $[a, b]$, with $a < b$.

Definition. The function f is *Riemann integrable* on $[a, b]$ with integral I if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every partition P with $\|P\| < \delta$, and for every choice of sample points $\xi_i \in [x_{i-1}, x_i]$,

$$\left| \sum_{i=1}^n f(\xi_i) \Delta x_i - I \right| < \varepsilon.$$

We write

$$I = \int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} S(f; P, \xi).$$

The last expression abbreviates the quantified definition; the limit must be the same for all sufficiently fine partitions and all sample points. The number δ may depend on f, a, b, ε , but not on the particular partition or its sample points.

A fixed choice of equal intervals or sample points is only one of the choices allowed by the definition. The requirement is that every sufficiently fine partition, with every choice of sample points, gives a sum within ε of the same number I .

The notation $\int_a^b f(x) dx$ is called the *definite integral* of f from a to b . Here f is the integrand, and a and b are the lower and upper limits of integration. For fixed f, a, b , the definite integral is a number. The integration variable may be renamed:

$$\int_a^b f(x) dx = \int_a^b f(t) dt.$$

For a continuous nonnegative function, this gives the area under its graph. For functions that take negative values, the sums retain the signs of those values.

Theorem. Every continuous real-valued function on a closed bounded interval $[a, b]$ is Riemann integrable on that interval.

This theorem was stated without proof. Continuity also ensures boundedness on $[a, b]$, by the Extreme Value Theorem, so a continuous function satisfies the boundedness assumption in the definition.