

1. Curve sketching

For a complete sketch, determine the domain, intercepts and any symmetry; find relevant limits and asymptotes; use f' to determine monotonicity and local extrema; use f'' to determine concavity and inflection points. Combine these facts in one graph, keeping separate components of the domain separate.

Example

Sketch

$$f(x) = \frac{x^3}{x^2 - 1}.$$

Domain, symmetry and intercept. The domain is $\mathbb{R} \setminus \{-1, 1\}$. Since $f(-x) = -f(x)$, the graph is symmetric about the origin. Its only intercept is $(0, 0)$.

Asymptotes. Polynomial division gives

$$f(x) = x + \frac{x}{x^2 - 1}.$$

Since $f(x) - x \rightarrow 0$ as $x \rightarrow \pm\infty$, the line $y = x$ is an oblique asymptote. There is no horizontal asymptote. The limits at the two excluded points are

x	-1^-	-1^+	1^-	1^+
$f(x)$	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Thus $x = -1$ and $x = 1$ are vertical asymptotes.

First derivative. Differentiating the divided expression and simplifying,

$$f'(x) = 1 - \frac{x^2 + 1}{(x^2 - 1)^2} = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2}.$$

The critical numbers are $-\sqrt{3}, 0, \sqrt{3}$. The excluded points ± 1 are not critical numbers. The denominator is positive throughout the domain, and the factor x^2 is positive except at 0. Therefore

Interval	Sign of f'	Behavior of f
$(-\infty, -\sqrt{3})$	+	increasing
$(-\sqrt{3}, -1)$	-	decreasing
$(-1, 0), (0, 1)$	-	decreasing
$(1, \sqrt{3})$	-	decreasing
$(\sqrt{3}, \infty)$	+	increasing

There is a strict local maximum at $(-\sqrt{3}, -3\sqrt{3}/2)$ and a strict local minimum at $(\sqrt{3}, 3\sqrt{3}/2)$. Neither is absolute on the domain. At 0, the derivative has the same negative sign on both sides, so there is no extremum. Continuity joins the two decreasing pieces into a strictly decreasing function on $(-1, 1)$.

1. Curve sketching (continued)

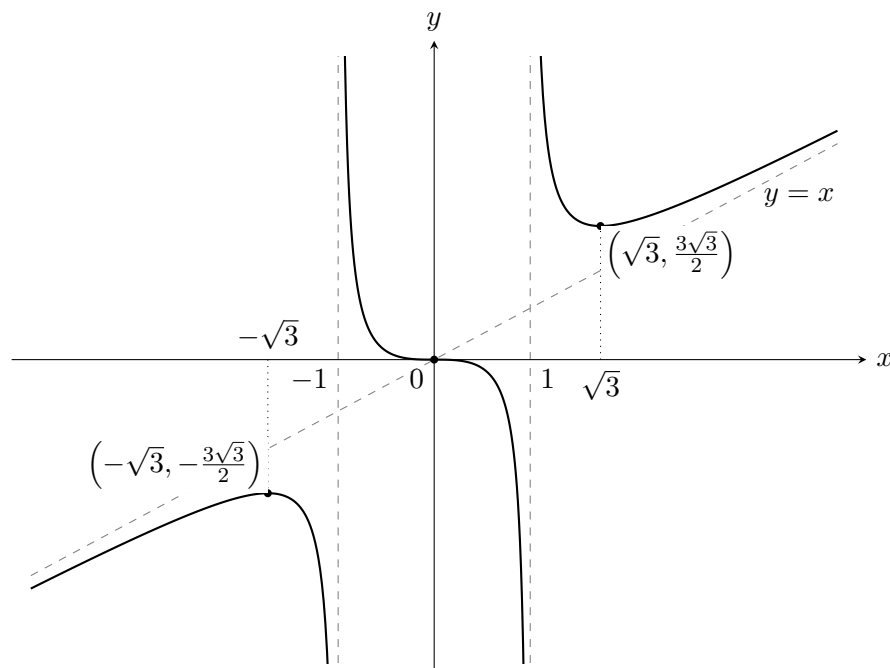
Second derivative. Using the expression $f'(x) = 1 - (x^2 + 1)/(x^2 - 1)^2$,

$$f''(x) = -\frac{2x}{(x^2 - 1)^2} + \frac{4x(x^2 + 1)}{(x^2 - 1)^3} = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

Because $x^2 + 3 > 0$, its sign is determined by x and $x^2 - 1$:

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f''(x)$	-	+	-	+
Concavity	downward	upward	downward	upward

The only inflection point is $(0, 0)$. Its tangent is $y = 0$. The points ± 1 are not in the domain, so they cannot give inflection points.



2. L'Hôpital's rule

A quotient whose numerator and denominator both tend to zero has *indeterminate form* $0/0$: the separate limits do not determine the quotient's limit. For example, as $x \rightarrow 0^+$,

$$\frac{x}{x} = 1, \quad \frac{x^2}{x} \rightarrow 0, \quad \frac{x}{x^2} \rightarrow +\infty.$$

Likewise, when both numerator and denominator tend to signed infinities, the form ∞/∞ alone gives no answer. These names describe limiting behavior; they are not arithmetic with zero or infinity.

Theorem (L'Hôpital's rule). Consider a one-sided limit $x \rightarrow a^+$ or $x \rightarrow a^-$, or a limit $x \rightarrow +\infty$ or $x \rightarrow -\infty$. Suppose that, sufficiently near the limiting point on the side in question (or sufficiently far along the relevant half-line):

- (i) f and g are differentiable, and $g(x) \neq 0$, $g'(x) \neq 0$;
- (ii) either $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$, or each of $f(x)$ and $g(x)$ tends to $+\infty$ or $-\infty$;
- (iii) the limit $f'(x)/g'(x) \rightarrow L$ exists, where L is a real number, $+\infty$, or $-\infty$.

Then, in the same limiting direction,

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$$

For a two-sided finite-point limit, apply the theorem on both sides and check that the results agree. No value or derivative at a is required. A proof of the finite-point $0/0$ case is given in Section 5.

Example

As $x \rightarrow 1$, both $\ln x$ and $x - 1$ tend to zero. They are differentiable near 1, and the denominator derivative is 1. Therefore

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \lim_{x \rightarrow 1} \frac{1/x}{1} = 1.$$

The theorem compares two limits. It does not assert that $f/g = f'/g'$, and f'/g' is not the derivative of f/g . In an application, differentiate the numerator and denominator separately.

3. Applications of the rule

Example

Find $\lim_{x \rightarrow 0} (e^x - 1 - x)/x^2$. The original form is $0/0$, and $(x^2)' = 2x \neq 0$ for $x \neq 0$. The first derivative quotient still has form $0/0$; its denominator derivative is 2 . A second application gives

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

Read the justification from right to left: the last limit exists, so the rule gives the middle limit, and then the original limit.

Example

As $x \rightarrow +\infty$, both $\ln x$ and x tend to $+\infty$, and the denominator derivative is 1 . Thus

$$\lim_{x \rightarrow +\infty} \frac{\ln x}{x} = \lim_{x \rightarrow +\infty} \frac{1/x}{1} = 0.$$

Both functions grow without bound, but their quotient tends to zero.

Example

For $x \rightarrow 0^+$, the quotient $\ln(1+x)/x^2$ has form $0/0$. On a sufficiently small interval with $x > 0$, the denominator derivative $2x$ is nonzero, and

$$\frac{(\ln(1+x))'}{(x^2)'} = \frac{1}{2x(1+x)} \rightarrow +\infty.$$

Hence $\lim_{x \rightarrow 0^+} \ln(1+x)/x^2 = +\infty$. The direction of approach matters: the left-hand limit is $-\infty$, by the same argument.

Checking the form. By continuity,

$$\lim_{x \rightarrow 0} \frac{x^2 + 1}{x + 1} = 1.$$

This has form $1/1$, so L'Hôpital's rule does not apply. The quotient of derivatives tends to 0 , a different answer.

Example

For $x \rightarrow +\infty$,

$$\frac{x + \sin x}{x} = 1 + \frac{\sin x}{x} \rightarrow 1$$

by $|\sin x| \leq 1$ and the squeeze theorem. The derivative quotient is $1 + \cos x$, which has no limit: at $2\pi n$ it equals 2 , and at $(2n+1)\pi$ it equals 0 . Thus a missing derivative-quotient limit does not establish that the original quotient has no limit.

4. Transformed limits

For a product of form $0 \cdot \infty$, rewrite it as a quotient before applying L'Hôpital's rule. A difference of form $\infty - \infty$ may be simplified by a common denominator or rationalization. For a positive base $u(x)$, use

$$u(x)^{v(x)} = \exp(v(x) \ln u(x)).$$

If $v(x) \ln u(x) \rightarrow L \in \mathbb{R}$, continuity of the exponential gives $u(x)^{v(x)} \rightarrow e^L$. This handles the forms 1^∞ and 0^0 after the logarithm has been analyzed.

Example

For $x \rightarrow 0^+$, $x \ln x$ has the product form $0 \cdot (-\infty)$. Rewrite it as a quotient:

$$x \ln x = \frac{\ln x}{1/x}.$$

The numerator tends to $-\infty$ and the denominator to $+\infty$. Both are differentiable for $x > 0$, with nonzero denominator derivative. Consequently

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

The power x^x has form 0^0 as $x \rightarrow 0^+$. Since $x^x = \exp(x \ln x)$ for $x > 0$, continuity of the exponential gives $\lim_{x \rightarrow 0^+} x^x = 1$.

Example

Find $\lim_{x \rightarrow 0} (1+x)^{1/x}$. The right-hand limit has form 1^∞ . The base is positive for $x > -1$. Take logarithms of the positive expression for $x \neq 0$:

$$\ln((1+x)^{1/x}) = \frac{\ln(1+x)}{x}.$$

This quotient has form $0/0$, and

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1}{1+x} = 1.$$

Exponentiating and using continuity yields

$$\boxed{\lim_{x \rightarrow 0} (1+x)^{1/x} = e.}$$

Here e and the exponential derivative were already introduced; this computation is a consequence of those results, not their proof.

Example

For $x > 0$, rationalization gives

$$\sqrt{x^2 + x} - x = \frac{x}{\sqrt{x^2 + x} + x} = \frac{1}{\sqrt{1 + 1/x} + 1}.$$

Thus $\lim_{x \rightarrow +\infty} (\sqrt{x^2 + x} - x) = 1/2$. The original difference has form $\infty - \infty$; algebra and continuity resolve it without L'Hôpital's rule.

5. A proof of the finite-point 0/0 case

We prove the finite-point 0/0 case using Rolle's theorem. The other versions of the rule are stated without proof.

Cauchy's Mean Value Theorem. Suppose f, g are continuous on $[u, v]$, differentiable on (u, v) , and $g'(t) \neq 0$ for every $t \in (u, v)$. Then there is $c \in (u, v)$ with

$$\frac{f(v) - f(u)}{g(v) - g(u)} = \frac{f'(c)}{g'(c)}.$$

Proof. Rolle's theorem shows $g(v) \neq g(u)$; otherwise g' would vanish somewhere. Set

$$A = \frac{f(v) - f(u)}{g(v) - g(u)}, \quad H(t) = f(t) - Ag(t).$$

The function H is continuous on $[u, v]$, differentiable on (u, v) , and $H(v) = H(u)$. Rolle gives $H'(c) = 0$, or $f'(c) = Ag'(c)$. Divide by the nonzero $g'(c)$. $\square$

Application to L'Hôpital's rule. Suppose f, g are differentiable on (a, b) , $g' \neq 0$ there, and

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x) = 0, \quad \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}.$$

Extend the functions by $f(a) = g(a) = 0$; they are then continuous at a . For every $x \in (a, b)$, Rolle ensures $g(x) \neq 0$, and Cauchy's theorem on $[a, x]$ gives some $c_x \in (a, x)$ with

$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}.$$

Given $\varepsilon > 0$, choose $\delta > 0$ small enough that $a + \delta < b$ and

$$0 < t - a < \delta \implies \left| \frac{f'(t)}{g'(t)} - L \right| < \varepsilon.$$

If $0 < x - a < \delta$, then $0 < c_x - a < x - a < \delta$, so

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c_x)}{g'(c_x)} - L \right| < \varepsilon.$$

This proves $f(x)/g(x) \rightarrow L$. For $L = +\infty$, replace the last absolute-value bound by $f'(t)/g'(t) > M$ for any prescribed $M > 0$; for $-\infty$, use a bound below $-M$. The left-hand argument uses $[x, a]$. Matching one-sided conclusions give the two-sided case. No derivative at a was needed. $\square$