

1. The first-derivative test

Recall that, on an interval, a positive derivative implies strict increase and a negative derivative implies strict decrease.

Theorem. Suppose f is continuous on an open interval containing c and differentiable there except possibly at c . Assume the indicated signs hold throughout sufficiently small intervals on the two sides of c .

Sign of f' on the left	Sign on the right	Conclusion at c
+	−	strict local maximum
−	+	strict local minimum
+	+	no local extremum
−	−	no local extremum

A strict local maximum means $f(x) < f(c)$ for all sufficiently close $x \neq c$; a strict local minimum uses $f(x) > f(c)$.

Proof. For nearby points $x < c < y$, MVT on $[x, c]$ and $[c, y]$ gives $\xi \in (x, c)$ and $\eta \in (c, y)$ such that

$$f(c) - f(x) = f'(\xi)(c - x), \quad f(y) - f(c) = f'(\eta)(y - c).$$

If the derivative is positive on the left and negative on the right, both $f(x)$ and $f(y)$ are less than $f(c)$. Reversing the signs proves the minimum case. If both signs are positive, then $f(x) < f(c) < f(y)$; if both are negative, then $f(x) > f(c) > f(y)$. These give no extremum. $\square$

Continuity at c is needed to apply MVT with c as an endpoint. The derivative at c need not exist.

Example

For $f(x) = x^3 - 3x$ on $\mathbb{R}$,

$$f'(x) = 3(x - 1)(x + 1), \quad \begin{array}{c|ccccc} x & (-\infty, -1) & -1 & (-1, 1) & 1 & (1, \infty) \\ \hline f'(x) & + & 0 & - & 0 & + \end{array}$$

There is a strict local maximum $f(-1) = 2$ and a strict local minimum $f(1) = -2$. Neither is absolute on $\mathbb{R}$, because the polynomial is unbounded above and below.

For x^3 , the derivative is positive on both sides of 0, so 0 is not an extremum. For $|x|$, the derivative changes from negative to positive at 0, giving a strict local minimum despite the missing derivative there.

2. Concavity

Let f be differentiable on an open interval I . At $a \in I$, its tangent line has equation

$$T_a(x) = f(a) + f'(a)(x - a).$$

Definition. The graph is *concave upward* on I if it lies above each tangent line away from the point of tangency:

$$f(x) > T_a(x) \quad (a, x \in I, x \neq a).$$

It is *concave downward* if the inequality is reversed. Here we use strict inequalities; a straight line has neither type of strict curvature.

Tangent slopes. If f' is strictly increasing on I , then f is concave upward. If f' is strictly decreasing, then f is concave downward.

Proof of the upward case. Fix $a \in I$. For $x > a$, MVT gives $\xi \in (a, x)$ with

$$f(x) - f(a) = f'(\xi)(x - a) > f'(a)(x - a).$$

For $x < a$, the same identity holds with $\xi \in (x, a)$. Now $f'(\xi) < f'(a)$ and $x - a < 0$, so multiplication again gives $f(x) - f(a) > f'(a)(x - a)$. This is the required inequality. Apply the result to $-f$ for the downward case. $\square$

Second derivative. The derivative of f' , when it exists, is denoted by $f'' = (f')'$. Suppose f is twice differentiable on I . Applying the monotonicity criterion to f' gives

$$f'' > 0 \text{ on } I \implies f \text{ concave upward on } I,$$

$$f'' < 0 \text{ on } I \implies f \text{ concave downward on } I.$$

Indeed, $f'' > 0$ makes f' strictly increasing, and the tangent argument applies. The negative case is analogous.

Example

For $f(x) = e^{-x}$, we have $f'(x) = -e^{-x} < 0$ and $f''(x) = e^{-x} > 0$. Thus f is decreasing and concave upward on $\mathbb{R}$. The sign of f' determines increase or decrease; the sign of f'' determines how the slope changes.

The tangent characterization also has a converse. For $a < b$, the strict tangent inequalities at a and b give

$$f'(a) < \frac{f(b) - f(a)}{b - a} < f'(b).$$

Thus concavity upward is equivalent to strict increase of f' under the stated convention. The condition $f'' > 0$ is sufficient, not necessary.

3. Inflection points

Definition. A point $(c, f(c))$ is an *inflection point* if f is continuous at c and the graph changes concavity there: concave upward on a sufficiently small interval on one side and concave downward on the other.

If f'' changes from positive to negative, or from negative to positive, across c , and f is continuous at c , the concavity criterion gives an inflection point. A zero or an undefined value of f'' alone does not give this conclusion.

Example

Let $f(x) = x^3$. Then $f''(x) = 6x$, which is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. The graph is concave downward on the first interval and upward on the second. It is continuous at 0, so $(0, 0)$ is an inflection point. The tangent there is horizontal, but there is no local extremum.

Example

Let $f(x) = x^4$. Although $f''(0) = 0$, we have $f''(x) = 12x^2 > 0$ on both sides of 0. There is no change of concavity and no inflection point. In fact, $f'(x) = 4x^3$ is strictly increasing on $\mathbb{R}$, so the graph is concave upward on the whole real line, including across 0.

Example

Let $f(x) = \sqrt[3]{x}$, the real cube root on $\mathbb{R}$. For $x \neq 0$,

$$f'(x) = \frac{1}{3(\sqrt[3]{x})^2}, \quad f''(x) = -\frac{2}{9(\sqrt[3]{x})^5}.$$

Thus $f'' > 0$ for $x < 0$ and $f'' < 0$ for $x > 0$. Continuity at 0 gives an inflection point $(0, 0)$, although neither $f'(0)$ nor $f''(0)$ exists as a finite derivative.

Example

For $f(x) = 1/x$, $f''(x) = 2/x^3$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. Nevertheless, there is no inflection point at 0: the function is not defined there. Its two branches are separated by a vertical asymptote.

4. The second-derivative test

Theorem. Suppose f is differentiable on an open interval containing c , $f'(c) = 0$, and $f''(c)$ exists. Then

$$\begin{array}{l|l} f''(c) > 0 & \text{strict local minimum at } c \\ f''(c) < 0 & \text{strict local maximum at } c \end{array}$$

If $f''(c) = 0$, the test gives no conclusion.

Proof. Suppose $A = f''(c) > 0$. By the definition of this derivative,

$$\lim_{h \rightarrow 0} \frac{f'(c+h) - f'(c)}{h} = A.$$

Use $\varepsilon = A/2$. For all sufficiently small nonzero h ,

$$\left| \frac{f'(c+h)}{h} - A \right| < \frac{A}{2}, \quad \text{so} \quad \frac{f'(c+h)}{h} > \frac{A}{2} > 0.$$

Consequently $f'(c+h) < 0$ when $h < 0$, and $f'(c+h) > 0$ when $h > 0$. Differentiability makes f continuous at c , so the first-derivative test gives a strict local minimum. For $f''(c) < 0$, apply this argument to $-f$. $\square$

The proof uses the existence of $f''(c)$, not continuity of f'' on a neighborhood.

Example

Classify the critical numbers of $f(x) = x^2 e^{-x}$ on $\mathbb{R}$. The product and chain rules give

$$f'(x) = e^{-x}x(2-x), \quad f''(x) = e^{-x}(x^2 - 4x + 2).$$

Since $e^{-x} > 0$, the critical numbers are 0 and 2. Now $f''(0) = 2 > 0$ and $f''(2) = -2e^{-2} < 0$. Hence $f(0) = 0$ is a strict local minimum and $f(2) = 4/e^2$ is a strict local maximum.

When the test is inconclusive. All three functions below satisfy $f'(0) = f''(0) = 0$:

$f(x)$	Behavior at 0
x^4	strict local minimum
$-x^4$	strict local maximum
x^3	no local extremum

Use the first-derivative test or compare function values in such cases.

5. Applications of the Mean Value Theorem

Bounds from a derivative. Let $a < b$. Suppose f is continuous on $[a, b]$, differentiable on (a, b) , and $m \leq f'(x) \leq M$ there. By MVT there is $c \in (a, b)$ with

$$f(b) - f(a) = f'(c)(b - a).$$

Because $b - a > 0$,

$$\boxed{m(b - a) \leq f(b) - f(a) \leq M(b - a)}.$$

In particular, $|f'| \leq K$ gives $|f(b) - f(a)| \leq K(b - a)$.

Example

For $t \geq 0$, prove

$$\frac{t}{1+t} \leq \ln(1+t) \leq t.$$

At $t = 0$, all quantities are zero. For $t > 0$, apply MVT to $\ln x$ on $[1, 1+t]$: for some $c \in (1, 1+t)$,

$$\ln(1+t) = \frac{t}{c}, \quad \frac{t}{1+t} < \frac{t}{c} < t.$$

The upper bound also follows from concavity: $\ln x$ is concave downward because $(\ln x)'' = -1/x^2 < 0$, so its graph lies below its tangent $y = x - 1$ at $x = 1$.

Example

Show that $x^3 + x - 1 = 0$ has exactly one real solution. The polynomial $p(x) = x^3 + x - 1$ is continuous, with $p(0) = -1$ and $p(1) = 1$. IVT gives a root in $(0, 1)$. Since $p'(x) = 3x^2 + 1 > 0$ on $\mathbb{R}$, the function is strictly increasing and has at most one root. Thus the root is unique.