

## 1. Fermat's theorem

**Recall.** A local maximum at an interior point  $c$  means that  $f(x) \leq f(c)$  for all  $x$  sufficiently close to  $c$ . For a local minimum the inequality is reversed. An absolute extremum compares  $f(c)$  with every value on the stated domain. As in the preceding notes, local extrema are considered only at interior points of the domain.

The Extreme Value Theorem says that a continuous function on a closed, bounded interval  $[a, b]$  attains an absolute maximum and an absolute minimum. It establishes their existence, without locating them.

**Fermat's theorem.** Suppose  $c$  is an interior point of the domain of  $f$ ,  $f$  has a local maximum or minimum at  $c$ , and  $f$  is differentiable at  $c$ . Then

$$f'(c) = 0.$$

**Proof.** Suppose first that  $f$  has a local maximum at  $c$ . For all sufficiently small  $h$ ,

$$f(c+h) - f(c) \leq 0.$$

Dividing by  $h$  gives different inequalities on the two sides:

$$\frac{f(c+h) - f(c)}{h} \leq 0 \quad (h > 0), \quad \frac{f(c+h) - f(c)}{h} \geq 0 \quad (h < 0).$$

Because the derivative exists, both one-sided limits equal  $f'(c)$ . Passing to the limits gives  $f'(c) \leq 0$  and  $f'(c) \geq 0$ . Thus  $f'(c) = 0$ .

If  $f$  has a local minimum at  $c$ , then  $-f$  has a local maximum there. The case just proved gives  $-f'(c) = 0$ .  $\square$

### Example

For  $f(x) = x^3$  on  $\mathbb{R}$ ,  $f'(0) = 0$ . Nevertheless,  $f(x) < f(0)$  for  $x < 0$  and  $f(x) > f(0)$  for  $x > 0$ . Thus 0 is neither a local maximum nor a local minimum. The converse of Fermat's theorem is false.

### Example

The function  $f(x) = |x|$  has a minimum at 0. Its left-hand derivative is  $-1$  and its right-hand derivative is  $1$ , so  $f'(0)$  does not exist. An extremum need not have a zero derivative unless differentiability is assumed.

For  $f(x) = x$  on  $[0, 1]$ , the absolute minimum and maximum occur at the endpoints, although the derivative of the formula is 1. Fermat's conclusion concerns interior points.

## 2. Critical numbers and absolute extrema

**Definition.** A number  $c$  in the domain of  $f$  is a *critical number* if  $f'(c) = 0$  or  $f'(c)$  does not exist. In a closed-interval extremum problem, list the interior critical numbers and then check the endpoints separately. A point outside the domain is not a critical number.

Fermat's theorem implies that every interior local extremum occurs at a critical number. Critical numbers are possible locations; the theorem does not classify them as maxima or minima.

**Closed-interval method.** Suppose  $f$  is continuous on  $[a, b]$ , where  $a < b$ . Then its absolute maximum and minimum occur either at an endpoint or at an interior critical number. When this gives a finite list:

1. Find every critical number in  $(a, b)$ , including points where the derivative does not exist.
2. Evaluate  $f$  at those numbers and at  $a$  and  $b$ .
3. Compare the values, recording all points where the largest and smallest values occur.

**Justification.** EVT guarantees that both extrema are attained. If an extremum is attained at an interior point, it is also local. Either the derivative fails to exist there, or Fermat's theorem makes it zero. This exhausts the possible locations. If the critical set is infinite, the same statement holds, but a finite list need not be available.

### Example

Find the absolute extrema of  $f(x) = x^3 - 3x$  on  $[-2, 2]$ . This polynomial is continuous on the whole interval, and

$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1).$$

The derivative exists everywhere; its zeros are  $-1$  and  $1$ . Both lie in the interior of the interval. The comparison is

|        |      |      |      |     |
|--------|------|------|------|-----|
| $x$    | $-2$ | $-1$ | $1$  | $2$ |
| $f(x)$ | $-2$ | $2$  | $-2$ | $2$ |

Hence the absolute maximum is  $2$ , attained at  $-1$  and  $2$ , and the absolute minimum is  $-2$ , attained at  $-2$  and  $1$ .

### 3. Rolle's theorem

**Theorem.** Let  $a < b$ . Suppose that

- (i)  $f$  is continuous on  $[a, b]$ ;
- (ii)  $f$  is differentiable on  $(a, b)$ ;
- (iii)  $f(a) = f(b)$ .

Then there is at least one  $c \in (a, b)$  such that

$$f'(c) = 0.$$

The graph has a horizontal tangent at some interior point.

**Proof.** By EVT,  $f$  attains an absolute minimum and an absolute maximum. If  $f$  is constant, its derivative is zero at every interior point. Otherwise, there is some  $x_0 \in (a, b)$  for which  $f(x_0) \neq f(a)$ .

If  $f(x_0) > f(a)$ , the absolute maximum is strictly greater than both endpoint values. A point  $c$  where it is attained must lie in  $(a, b)$ . It is a local maximum, and  $f$  is differentiable there, so Fermat's theorem gives  $f'(c) = 0$ . If  $f(x_0) < f(a)$ , the same argument uses an interior absolute minimum.  $\square$

#### Example

For  $f(x) = x^2 - 4x$  on  $[0, 4]$ , continuity and differentiability follow because  $f$  is a polynomial, and  $f(0) = f(4) = 0$ . Since  $f'(x) = 2x - 4$ , the point supplied by Rolle's theorem is  $c = 2 \in (0, 4)$ .

**Hypotheses.** For  $f(x) = |x|$  on  $[-1, 1]$ , continuity and equal endpoint values hold. However, the derivative is  $-1$  for  $x < 0$ ,  $1$  for  $x > 0$ , and undefined at  $0$ . There is no point where the derivative is zero: differentiability in the open interval cannot be omitted.

For

$$f(x) = \begin{cases} x, & 0 \leq x < 1, \\ 0, & x = 1, \end{cases}$$

the endpoint values agree and  $f'(x) = 1$  on  $(0, 1)$ . Continuity fails at  $1$ , so Rolle's theorem does not apply. Continuity is required at the endpoints as well as inside.

Finally,  $f(x) = x$  on  $[0, 1]$  satisfies continuity and differentiability but not equal endpoint values. Its derivative never vanishes.

## 4. The Mean Value Theorem

**Theorem.** Let  $a < b$ . If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , there is at least one  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

At an interior point, a tangent is parallel to the secant joining the endpoints. The theorem does not require a derivative at either endpoint and does not assert that  $c$  is unique or the midpoint.

**Proof.** Let  $m = [f(b) - f(a)]/(b - a)$  and subtract the secant line:

$$g(x) = f(x) - [f(a) + m(x - a)].$$

Then  $g$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and  $g(a) = g(b) = 0$ . Rolle's theorem supplies  $c \in (a, b)$  with

$$0 = g'(c) = f'(c) - m.$$

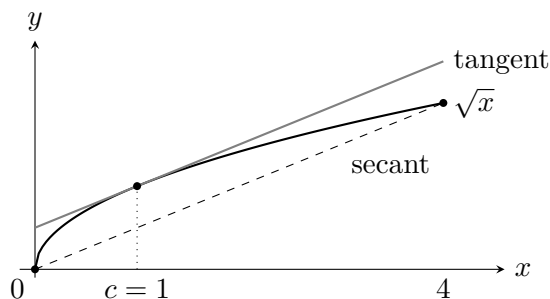
Thus  $f'(c) = m$ , as required.  $\square$

### Example

Apply MVT to  $f(x) = \sqrt{x}$  on  $[0, 4]$ . It is continuous on this closed interval and differentiable on  $(0, 4)$ . The secant slope is  $(2 - 0)/(4 - 0) = 1/2$ , so

$$\frac{1}{2\sqrt{c}} = \frac{1}{2} \implies c = 1.$$

The derivative at 0 does not exist as a finite right derivative; this does not affect the hypotheses.



## 5. Consequences of the Mean Value Theorem

Throughout this section,  $I$  is an interval,  $f$  is continuous on  $I$ , and  $f$  is differentiable at every interior point of  $I$ . For any  $x < y$  in  $I$ , the entire interval  $[x, y]$  lies in  $I$ , so MVT gives  $c \in (x, y)$  with

$$f(y) - f(x) = f'(c)(y - x).$$

**Zero derivative.** If  $f'(x) = 0$  throughout the interior of  $I$ , then  $f$  is constant on  $I$ . **Proof:** the displayed identity gives  $f(y) - f(x) = 0$  for every  $x < y$  in  $I$ .  $\square$

If two functions  $f, g$  satisfy the same hypotheses and  $f' = g'$  on the interior, then

$$f(x) = g(x) + C \quad \text{on } I$$

for a constant  $C$ : apply the result to  $f - g$ .

### Example

If  $F'(x) = 2x$  for all  $x \in \mathbb{R}$  and  $F(1) = 5$ , then  $(F - x^2)' = 0$ . Hence  $F(x) = x^2 + C$ ; substituting  $x = 1$  gives  $C = 4$ , so  $F(x) = x^2 + 4$ .

**Monotonicity.** A function is *strictly increasing* on  $I$  if  $x < y$  implies  $f(x) < f(y)$ , and *nondecreasing* if  $x < y$  implies  $f(x) \leq f(y)$ . Strictly decreasing and nonincreasing use the reversed inequalities. Under the hypotheses above:

| Derivative on the interior of $I$ | Conclusion on $I$   |
|-----------------------------------|---------------------|
| $f' > 0$                          | strictly increasing |
| $f' < 0$                          | strictly decreasing |
| $f' \geq 0$                       | nondecreasing       |
| $f' \leq 0$                       | nonincreasing       |

**Proof.** In the displayed MVT identity,  $y - x > 0$ . Thus the sign of  $f(y) - f(x)$  is the sign of  $f'(c)$ , giving all four statements.  $\square$

Strict increase does not require  $f' > 0$  at every point:  $x^3$  is strictly increasing on  $\mathbb{R}$ , although its derivative vanishes at 0.

**The interval condition.** On  $\mathbb{R} \setminus \{0\}$ , define  $f(x) = 0$  for  $x < 0$  and  $f(x) = 1$  for  $x > 0$ . Its derivative is zero everywhere in its domain, but it is not constant on that domain. It is constant on each of the intervals  $(-\infty, 0)$  and  $(0, \infty)$ . An interval crossing the missing point cannot be used in MVT.