

Homework 12

For Monday, November 2, 2026.

Give exact answers and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. Express antiderivatives as functions F with the required derivative. For questions about Riemann integrals, use the definition or the theorem that continuous functions on closed bounded intervals are integrable.

1. Find all antiderivatives on $(0, \infty)$ of

$$f(x) = 3\sqrt{x} - \frac{4}{x^2} + 2.$$

Check your answer by differentiation.

2. On $(0, \infty)$, consider

$$F(x) = x \ln x - x, \quad G(x) = x \ln x.$$

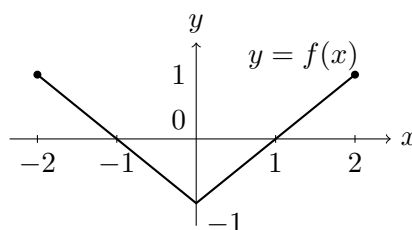
Determine which of these functions is an antiderivative of $\ln x$. Then describe all antiderivatives of $\ln x$ on this interval.

3. Find F on $(-3, 3)$ such that

$$F'(x) = \frac{1}{\sqrt{9-x^2}}, \quad F(0) = \frac{\pi}{6}.$$

Verify both conditions and explain why they determine F uniquely.

4. A particle has position $x(t)$ and velocity $v(t) = 3(t-1)^2 - 2$, with $x(0) = 5$. Find $x(t)$ and the change in position $x(2) - x(0)$. Verify your position formula by differentiation.
5. The graph below defines f on $[-2, 2]$. Let F satisfy $F' = f$ on $(-2, 2)$ and $F(0) = 0$. Without finding a formula for F , determine its maximal open intervals of increase and decrease, the x -coordinates and types of its interior local extrema, its concavity intervals, and its inflection points.



6. Find all differentiable functions F on $\mathbb{R} \setminus \{0\}$ such that

$$F'(x) = 2x, \quad F(-1) = 1, \quad F(1) = 4.$$

Can any such function be extended continuously to $x = 0$? Explain.

7. Suppose $f(x) > 0$ for every real x , and $F' = f$ on $\mathbb{R}$. Prove that F cannot be periodic. Here periodic means that there exists $T > 0$ with $F(x + T) = F(x)$ for every x . Then give an example of a periodic function g , not identically zero, that does have a periodic antiderivative.

8. Does the function

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0 \end{cases}$$

have an antiderivative on $\mathbb{R}$? Prove your answer. You may begin by determining the possible formulas on the two half-lines, then use continuity and the definition of the derivative at 0.

9. For the partition

$$P: \quad -1 < -\frac{1}{2} < 1 < 2,$$

find all subinterval widths and the mesh. Which of the two lists

$$(\xi_1, \xi_2, \xi_3) = \left(-\frac{3}{4}, 0, \frac{3}{2}\right), \quad (\eta_1, \eta_2, \eta_3) = \left(-\frac{3}{4}, \frac{3}{2}, \frac{3}{2}\right)$$

is a valid choice of sample points for P ? Justify your answer.

10. A particle's continuous velocity has the following measured values. Time is in seconds and velocity is in metres per second.

t	0	1	3	4
$v(t)$	2	-1	-2	1

Use left endpoints on the intervals $[0, 1]$, $[1, 3]$, and $[3, 4]$ to estimate $x(4) - x(0)$. Include units. Does the table determine the exact change in position? Explain.

11. Let A be the area under $f(x) = 1/(1+x)$ on $[0, 2]$. Use four equal subintervals to obtain a lower rectangle estimate and an upper rectangle estimate for A . Calculate both exactly, and explain which uses left endpoints and which uses right endpoints.
12. Express the following limit as a definite integral, without evaluating it:

$$\lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \sqrt{1 + \left(2 + \frac{3i}{n}\right)^2}.$$

Give the interval, integrand, subinterval width and sample points in your representation, and explain why the limit exists.

13. For $f(x) = \cos x$ on $[0, \pi]$, write the midpoint Riemann sum M_n for n equal subintervals. Use $\cos(\pi - t) = -\cos t$ to show that $M_n = 0$ for every positive integer n , including odd n . Deduce the value of $\int_0^\pi \cos x \, dx$ from the definition and continuous integrability.
14. For each positive integer n , construct a partition P_n of $[0, 1]$ with $n+1$ subintervals and mesh $1/2$. Explain why an increasing number of subintervals alone does not meet the condition on partitions in the definition of the Riemann integral.
15. Let f be bounded on $[1, 3]$. Write the full epsilon-delta statement that f is Riemann integrable with $\int_1^3 f(x) \, dx = 2$, including the conditions on partitions and sample points. Then write the negation of that statement.

- 16.** For each function below on $[-1, 1]$, decide whether the theorem that continuous functions are Riemann integrable applies. Explain your answer; do not calculate an integral.

$$(a) f(x) = \sqrt{1 - x^2}, \quad (b) g(x) = \begin{cases} \sin x/x, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

$$(c) h(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

If the theorem does not apply, state whether that alone proves the function is not integrable.

- 17.** Let $k \in \mathbb{R}$ and $a < b$. Use the definition of Riemann integrability to prove that the constant function $f(x) = k$ is integrable on $[a, b]$, and determine its integral.

- 18.** On $[0, 1]$, define

$$d(x) = \begin{cases} 1, & x \text{ is rational,} \\ 0, & x \text{ is irrational.} \end{cases}$$

Prove directly from the definition that d is not Riemann integrable. You may use the fact that every nonempty open interval contains both rational and irrational numbers.

- 19.** Let $P : 0 = x_0 < x_1 < \cdots < x_n = 1$, with sample points $\xi_i \in [x_{i-1}, x_i]$ and widths $\Delta x_i = x_i - x_{i-1}$. Prove that

$$\left| \sum_{i=1}^n \xi_i \Delta x_i - \frac{1}{2} \right| \leq \frac{1}{2} \|P\|.$$

Use this estimate to prove directly that $\int_0^1 x \, dx = 1/2$, giving a choice of δ for each $\varepsilon > 0$. You may use $x_i^2 - x_{i-1}^2 = (x_i + x_{i-1})\Delta x_i$.

- 20.** Define p on $[0, 1]$ by

$$p(x) = \begin{cases} 1, & x = 1/2, \\ 0, & x \neq 1/2. \end{cases}$$

Prove directly from the definition that p is Riemann integrable with integral 0, even though it is discontinuous. For an arbitrary partition, how many subintervals can have $1/2$ as a sample point? Use this to bound every Riemann sum in terms of the mesh.