

Homework 11

For Thursday, October 22, 2026.

Give exact answers and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. Whenever you use L'Hôpital's rule, check its hypotheses at each application. Use the strict tangent-line definition of concavity from Lecture 11.

1. Sketch $f(x) = x + 1/x$ on its natural domain. Determine the intercepts, symmetry, vertical and oblique asymptotes, maximal open intervals of increase and decrease, local extrema, concavity intervals, and any inflection points. State the range and label the significant features on your sketch.
2. Sketch $f(x) = xe^{-x}$ on $\mathbb{R}$. Determine the intercepts, limits at both ends of the real line, horizontal asymptotes, intervals of increase and decrease, local and absolute extrema, concavity intervals, and inflection points. Label the significant features on your sketch.
3. For $f(x) = (x^2 + x)/(x^2 + 1)$, find the horizontal asymptote and every point where the graph meets it. Determine where the graph lies above and below that line.
4. Let $f(x) = (x^2 - 1)/(x^2 - x)$ on its natural domain. Identify the excluded points and determine which discontinuities can be removed by assigning a value. Give each required value and find all vertical and horizontal asymptotes.

5. Evaluate

$$\lim_{x \rightarrow 0} \frac{\ln(1 + \sin x)}{x}.$$

6. Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}.$$

7. Evaluate

$$\lim_{x \rightarrow +\infty} \frac{x^2}{e^x}.$$

8. Find the left-hand and right-hand limits at 0 of $f(x) = |x|/\sin x$. Does the two-sided limit exist?

9. Evaluate

$$\lim_{x \rightarrow 0^+} \sqrt{x} \ln x.$$

10. Evaluate

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right).$$

11. Evaluate

$$\lim_{x \rightarrow 0} (\cos x)^{1/x^2}.$$

12. Evaluate $\lim_{x \rightarrow 0^+} x^{1/\ln x}$. Identify the limiting form of the base and exponent.

13. For $a, b \in \mathbb{R}$, define F on $(-1, \infty)$ by

$$F(x) = \begin{cases} \frac{\ln(1+x) - ax}{x^2}, & x \neq 0, \\ b, & x = 0. \end{cases}$$

Find all a, b for which F is continuous at 0. For those values, determine whether F is differentiable at 0, and find $F'(0)$ if it exists.

14. For $x > 0$, let $f(x) = x + x^2 \sin(1/x)$ and $g(x) = x$. Find $\lim_{x \rightarrow 0^+} f(x)/g(x)$, and prove that $\lim_{x \rightarrow 0^+} f'(x)/g'(x)$ does not exist. Explain why this does not contradict L'Hôpital's rule.

15. Evaluate

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x}$$

in two ways: first using trigonometric identities and the known limit $\sin x/x \rightarrow 1$, and then using L'Hôpital's rule.

16. Apply Cauchy's Mean Value Theorem to $f(x) = x^2$ and $g(x) = x^3$ on $[1, 2]$. Find every $c \in (1, 2)$ for which

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}.$$

17. Suppose $a < b$, f, g are continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) > 0$ there. If constants m, M satisfy

$$m \leq \frac{f'(x)}{g'(x)} \leq M \quad (a < x < b),$$

prove that

$$m(g(b) - g(a)) \leq f(b) - f(a) \leq M(g(b) - g(a)).$$

18. Let f, g be defined on an open interval containing a , and suppose they are differentiable at a , with $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Prove directly from the definition of the derivative that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Include a justification that $g(x) \neq 0$ for all sufficiently close $x \neq a$. Do not assume the derivatives exist away from a .

19. Let f be differentiable on $(0, \infty)$, and suppose $\lim_{x \rightarrow +\infty} f(x) = L \in \mathbb{R}$. If $\lim_{x \rightarrow +\infty} f'(x) = m \in \mathbb{R}$ also exists, prove that $m = 0$.

20. Find the oblique asymptote as $x \rightarrow +\infty$ of

$$f(x) = x^2 \ln \left(1 + \frac{1}{x} \right), \quad x > 0.$$

Determine its slope and intercept using limits, and verify the asymptote by showing that the difference tends to zero.