

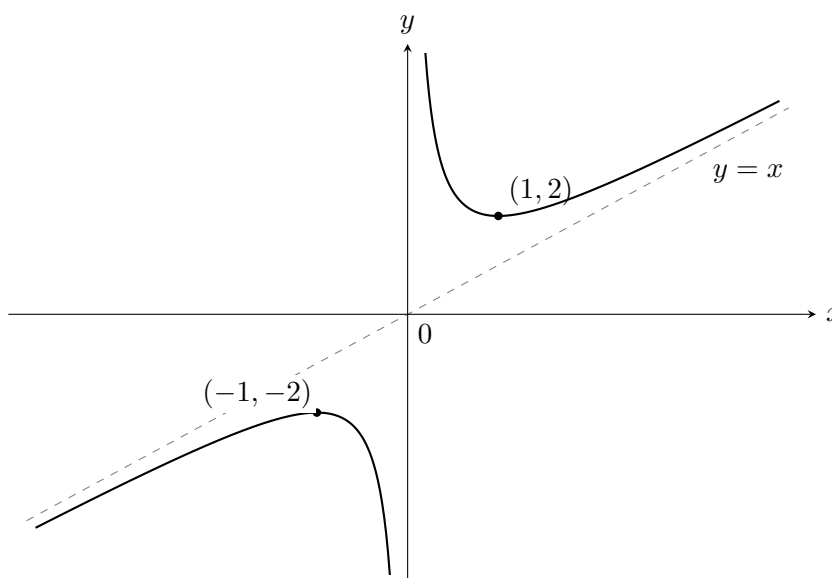
Homework 11: Solutions

Give exact answers and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. Whenever you use L'Hôpital's rule, check its hypotheses at each application. Use the strict tangent-line definition of concavity from Lecture 11.

1. The domain is $\mathbb{R} \setminus \{0\}$. The function is odd, with no intercepts: $f(x) = 0$ would require $x^2 + 1 = 0$, and 0 is outside the domain. Since $f(x) - x = 1/x \rightarrow 0$ as $x \rightarrow \pm\infty$, $y = x$ is an oblique asymptote. At 0, the left-hand limit is $-\infty$ and the right-hand limit is $+\infty$; thus $x = 0$ is a vertical asymptote.

$$f'(x) = 1 - \frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}.$$

The function increases on $(-\infty, -1)$ and $(1, \infty)$, and decreases on $(-1, 0)$ and $(0, 1)$. Its local maximum is $(-1, -2)$, and its local minimum is $(1, 2)$. It is concave downward on $(-\infty, 0)$ and upward on $(0, \infty)$. There is no inflection point: 0 is not in the domain. The negative branch has range $(-\infty, -2]$, and the positive branch has range $[2, \infty)$, by monotonicity, continuity and the endpoint limits. Hence the full range is $(-\infty, -2] \cup [2, \infty)$.



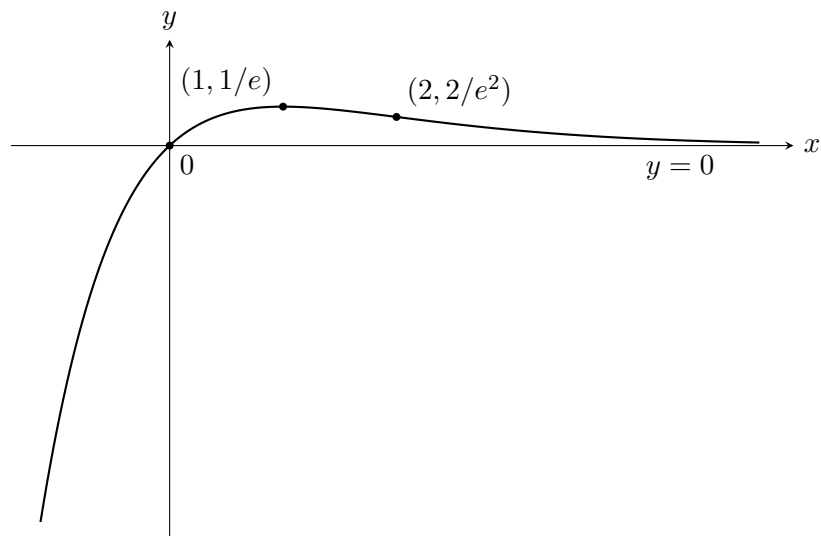
2. The domain is $\mathbb{R}$, and the only intercept is $(0, 0)$. At $+\infty$, the quotient x/e^x has form ∞/∞ , with denominator derivative $e^x > 0$. L'Hôpital's rule gives

$$\lim_{x \rightarrow +\infty} xe^{-x} = \lim_{x \rightarrow +\infty} \frac{1}{e^x} = 0.$$

For $x = -u$, $u \rightarrow +\infty$, the value is $-ue^u \rightarrow -\infty$. The only horizontal asymptote is $y = 0$, at $+\infty$.

$$f'(x) = (1 - x)e^{-x}, \quad f''(x) = (x - 2)e^{-x}.$$

Thus f increases on $(-\infty, 1)$ and decreases on $(1, \infty)$. The point $\boxed{(1, 1/e)}$ is a strict local and absolute maximum. There is no local minimum and no absolute minimum. The graph is concave downward on $(-\infty, 2)$ and upward on $(2, \infty)$. Continuity and the change of concavity give the unique inflection point $\boxed{(2, 2/e^2)}$.



3. Dividing numerator and denominator by x^2 gives $f(x) \rightarrow 1$ at both infinities, so the horizontal asymptote is $y = 1$. Furthermore,

$$f(x) - 1 = \frac{x - 1}{x^2 + 1}.$$

The denominator is positive. The graph is below the line for $x < 1$, meets it only at $(1, 1)$, and is above it for $x > 1$. An asymptote may be crossed.

4. The original denominator is $x(x - 1)$, so the domain excludes $0, 1$. For permitted x , cancellation gives

$$f(x) = \frac{(x - 1)(x + 1)}{x(x - 1)} = 1 + \frac{1}{x}.$$

At 1, the limit is 2; assigning $f(1) = 2$ removes that discontinuity. At 0, the one-sided limits are $-\infty$ and $+\infty$, so no finite assignment can make it continuous. The only vertical asymptote is $x = 0$. The only horizontal asymptote is $y = 1$, at both infinities. Cancellation does not include 1 in the original domain.

5. For all sufficiently small $|x|$, $1 + \sin x > 0$. The quotient has form $0/0$, both functions are differentiable, and the denominator derivative is 1. Therefore

$$\lim_{x \rightarrow 0} \frac{\ln(1 + \sin x)}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1 + \sin x} = 1.$$

6. Each of the first three quotients below has form $0/0$. The successive denominator derivatives are $3x^2, 6x, 6$, nonzero on a sufficiently small punctured interval. All functions involved are differentiable there. Repeated application of the rule gives

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{6x} = \lim_{x \rightarrow 0} \frac{-\cos x}{6} = \boxed{-\frac{1}{6}}.$$

The final limit exists, which justifies the preceding applications successively from right to left.

7. The first two quotients have form ∞/∞ as $x \rightarrow +\infty$. Their denominator derivatives are $e^x > 0$, and all functions are differentiable. Thus

$$\lim_{x \rightarrow +\infty} \frac{x^2}{e^x} = \lim_{x \rightarrow +\infty} \frac{2x}{e^x} = \lim_{x \rightarrow +\infty} \frac{2}{e^x} = \boxed{0}.$$

8. For $x > 0$, the numerator is x ; for $x < 0$, it is $-x$. Each one-sided quotient has form $0/0$, with denominator derivative $\cos x \neq 0$ near 0. Hence

$$\lim_{x \rightarrow 0^+} \frac{|x|}{\sin x} = \lim_{x \rightarrow 0^+} \frac{1}{\cos x} = 1, \quad \lim_{x \rightarrow 0^-} \frac{|x|}{\sin x} = \lim_{x \rightarrow 0^-} \frac{-1}{\cos x} = -1.$$

The two-sided limit does not exist, since these one-sided limits differ.

9. Rewrite the product for $x > 0$ as

$$\sqrt{x} \ln x = \frac{\ln x}{x^{-1/2}}.$$

The numerator tends to $-\infty$, the denominator to $+\infty$, and the denominator derivative $-\frac{1}{2}x^{-3/2}$ is nonzero. The rule gives

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x \rightarrow 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = \lim_{x \rightarrow 0^+} (-2\sqrt{x}) = \boxed{0}.$$

10. Combine the fractions first:

$$\frac{1}{e^x - 1} - \frac{1}{x} = \frac{x - e^x + 1}{x(e^x - 1)}.$$

This is $0/0$ as $x \rightarrow 0^+$. The first denominator derivative, $e^x - 1 + xe^x$, is positive for $x > 0$. The resulting quotient is still $0/0$, and its denominator derivative $(2 + x)e^x$ is also positive. All functions are differentiable for $x > 0$. Therefore

$$\lim_{x \rightarrow 0^+} \frac{x - e^x + 1}{x(e^x - 1)} = \lim_{x \rightarrow 0^+} \frac{1 - e^x}{e^x - 1 + xe^x} = \lim_{x \rightarrow 0^+} \frac{-e^x}{(2 + x)e^x} = \boxed{-\frac{1}{2}}.$$

11. For $0 < |x| < \pi/2$, the base is positive. Write

$$(\cos x)^{1/x^2} = \exp\left(\frac{\ln(\cos x)}{x^2}\right).$$

The logarithmic quotient and the first derivative quotient both have form $0/0$. The denominator derivatives $2x$ and 2 are nonzero on a punctured interval. Hence

$$\lim_{x \rightarrow 0} \frac{\ln(\cos x)}{x^2} = \lim_{x \rightarrow 0} \frac{-\tan x}{2x} = \lim_{x \rightarrow 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}.$$

By continuity of the exponential, the original limit is $\boxed{e^{-1/2}}$.

12. As $x \rightarrow 0^+$, the base tends to 0 and $1/\ln x \rightarrow 0$ from below: the limiting form is $\boxed{0^0}$. For every $0 < x < 1$, the exact power identity gives

$$x^{1/\ln x} = \exp\left(\frac{\ln x}{\ln x}\right) = e.$$

Thus the limit is $\boxed{e}$; L'Hôpital's rule is unnecessary.

13. If F is continuous at 0, it has a finite limit there, so $xF(x) \rightarrow 0$. But for $x \neq 0$,

$$xF(x) = \frac{\ln(1+x)}{x} - a \rightarrow 1 - a.$$

The displayed logarithmic limit is $0/0$; differentiation gives $1/(1+x) \rightarrow 1$, with denominator derivative 1. Consequently $a = 1$ is necessary. For this choice, another $0/0$ application has denominator derivative $2x \neq 0$:

$$\lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x^2} = \lim_{x \rightarrow 0} \frac{1/(1+x) - 1}{2x} = \lim_{x \rightarrow 0} \frac{-1}{2(1+x)} = -\frac{1}{2}.$$

Thus continuity holds exactly for $\boxed{a = 1, b = -1/2}$. For these parameters the difference quotient at 0 is

$$\frac{F(h) - F(0)}{h} = \frac{\ln(1+h) - h + h^2/2}{h^3}.$$

This has form $0/0$, with denominator derivative $3h^2 \neq 0$ for $h \neq 0$. The numerator derivative simplifies to $1/(1+h) - 1 + h = h^2/(1+h)$. Therefore

$$F'(0) = \lim_{h \rightarrow 0} \frac{h^2/(1+h)}{3h^2} = \lim_{h \rightarrow 0} \frac{1}{3(1+h)} = \boxed{\frac{1}{3}}.$$

All logarithms and derivatives used here are defined near 0 within $(-1, \infty)$.

14. For $x > 0$,

$$\frac{f(x)}{g(x)} = 1 + x \sin(1/x).$$

Since $|x \sin(1/x)| \leq x$, the squeeze theorem gives the limit $\boxed{1}$. Differentiation gives

$$\frac{f'(x)}{g'(x)} = 1 + 2x \sin(1/x) - \cos(1/x).$$

At $x = 1/(2\pi n)$ this equals 0; at $x = 1/((2n+1)\pi)$ it equals 2, for positive integers n . These points lie arbitrarily close to 0, so no limit exists. The derivative-quotient limit is a hypothesis of L'Hôpital's rule. Its failure gives no conclusion about the original quotient.

15. For small $x \neq 0$, rationalizing the numerator gives

$$\frac{1 - \cos x}{x \sin x} = \frac{\sin x}{x(1 + \cos x)} \rightarrow \boxed{\frac{1}{2}}.$$

Alternatively, two $0/0$ applications give

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x} = \lim_{x \rightarrow 0} \frac{\sin x}{\sin x + x \cos x} = \lim_{x \rightarrow 0} \frac{\cos x}{2 \cos x - x \sin x} = \frac{1}{2}.$$

The first denominator derivative is nonzero near 0, excluding 0, because $(\sin x + x \cos x)/x \rightarrow 2$. The next denominator derivative tends to 2, so it too is nonzero near 0. All functions are differentiable there.

16. Both polynomials are continuous on $[1, 2]$ and differentiable inside it, and $g'(x) = 3x^2 > 0$. Cauchy's theorem applies. Its equation becomes

$$\frac{4-1}{8-1} = \frac{2c}{3c^2}, \quad \frac{3}{7} = \frac{2}{3c}.$$

The unique solution is $\boxed{c = 14/9}$, which lies in $(1, 2)$.

17. MVT applied to g gives $g(b) - g(a) > 0$, since $g' > 0$. Cauchy's Mean Value Theorem supplies $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

The hypotheses give $m \leq f'(c)/g'(c) \leq M$. Multiplying by the positive number $g(b) - g(a)$ proves both inequalities. Continuity and differentiability are exactly those needed for the two Mean Value Theorem applications.

18. For $x \neq a$, set

$$A(x) = \frac{f(x)}{x-a}, \quad B(x) = \frac{g(x)}{x-a}.$$

Since $f(a) = g(a) = 0$, differentiability at a gives $A(x) \rightarrow f'(a)$ and $B(x) \rightarrow g'(a) \neq 0$. Choose $\delta > 0$ so that

$$0 < |x-a| < \delta \implies |B(x) - g'(a)| < \frac{|g'(a)|}{2}.$$

Then $|B(x)| > |g'(a)|/2 > 0$, so $g(x) = (x-a)B(x) \neq 0$ there. The quotient limit law now yields

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{A(x)}{B(x)} = \boxed{\frac{f'(a)}{g'(a)}}.$$

No derivative away from a was used.

19. Suppose first that $m > 0$. There is $R > 0$ such that $f'(t) > m/2$ for every $t > R$. For each $x > R$, differentiability implies continuity on $[R, x]$, so MVT gives $c \in (R, x)$ with

$$f(x) - f(R) = f'(c)(x - R) > \frac{m}{2}(x - R).$$

The right-hand side tends to $+\infty$, contradicting the finite limit of f . If $m < 0$, apply the same argument to $-f$, whose derivative tends to $-m > 0$ and whose value tends to $-L$. Both nonzero possibilities are excluded; hence $\boxed{m = 0}$.

20. If $y = mx + b$ is an oblique asymptote, dividing $f(x) - mx - b \rightarrow 0$ by $x \rightarrow +\infty$ gives $m = \lim f(x)/x$. Put $t = 1/x$; then $t \rightarrow 0^+$, and

$$m = \lim_{x \rightarrow +\infty} x \ln(1 + 1/x) = \lim_{t \rightarrow 0^+} \frac{\ln(1 + t)}{t} = 1.$$

Here the form is $0/0$, and the derivative quotient is $1/(1+t) \rightarrow 1$, with denominator derivative 1. The intercept must therefore be

$$\begin{aligned} b &= \lim_{x \rightarrow +\infty} (f(x) - x) = \lim_{t \rightarrow 0^+} \frac{\ln(1 + t) - t}{t^2} \\ &= \lim_{t \rightarrow 0^+} \frac{1/(1+t) - 1}{2t} = \lim_{t \rightarrow 0^+} \frac{-1}{2(1+t)} = -\frac{1}{2}. \end{aligned}$$

This is another $0/0$ application on $t > 0$, with denominator derivative $2t \neq 0$. Thus the asymptote is $\boxed{y = x - 1/2}$, and the computed intercept verifies

$$\lim_{x \rightarrow +\infty} [f(x) - (x - 1/2)] = b + \frac{1}{2} = 0.$$