

Homework 10

For Monday, October 19, 2026.

Give exact answers and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. Local extrema are considered only at interior points of the stated domain. Use the strict tangent-line definition of concavity from Lecture 11.

1. A differentiable function f on $\mathbb{R}$ satisfies

$$f'(x) = (x+2)(x-1)^2(x-3).$$

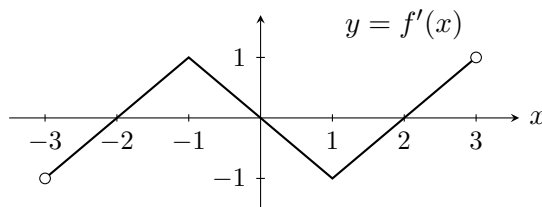
Find the maximal open intervals on which f is strictly increasing or strictly decreasing. Classify each critical number as a local maximum, a local minimum, or neither.

2. Find all critical numbers of $f(x) = |x^2 - 4|$ on $\mathbb{R}$, and classify each using the first-derivative test. Give the value of f at each local extremum. Include points where the derivative does not exist.
3. For $f(x) = \ln x/x$, with $x > 0$, find every critical number and use the second-derivative test to classify it. Give the value of f there.
4. Let $f(x) = \sin x - x$. Determine what the second-derivative test says at 0. Then use the first-derivative test to decide whether 0 is a local maximum, a local minimum, or neither.
5. Let

$$f(x) = \begin{cases} x^2, & x < 0, \\ -x^2, & x \geq 0. \end{cases}$$

Find $f'(0)$ from the definition and determine whether $f''(0)$ exists. Is $(0, 0)$ an inflection point? Is 0 a local extremum? Justify both conclusions.

6. Determine the maximal open intervals of concavity and all inflection points of $f(x) = \ln(1 + x^2)$. Give the coordinates of each inflection point.
7. A function f is differentiable on $(-3, 3)$. The graph of f' consists of the straight segments shown below.



Find the maximal open intervals of increase, decrease and concavity of f . Give the x -coordinates of every local maximum, local minimum and inflection point. A formula for f is not required.

8. (a) Use concavity and the tangent line to $\sqrt{x}$ at 1 to prove

$$\sqrt{x} \leq \frac{x+1}{2} \quad (x > 0).$$

Determine when equality holds.

- (b) Deduce that $\sqrt{uv} \leq (u+v)/2$ for $u, v > 0$, and determine when equality holds.

9. Suppose f is twice differentiable on $\mathbb{R}$, with $f'(x) > 0$ and $f''(x) > 0$ for every x . Let

$$g(x) = -3f(2-x) + 5.$$

Find g' and g'' . Determine whether g is increasing or decreasing, and whether it is concave upward or downward.

10. For $a \in \mathbb{R}$, let $f_a(x) = x^4 + ax^2$. Determine all values of a for which f_a is concave upward on the whole real line. Treat $a = 0$ explicitly.

11. Find $a, b, c \in \mathbb{R}$ so that $f(x) = x^3 + ax^2 + bx + c$ satisfies

$$f(1) = 2, \quad f'(1) = 3, \quad f''(1) = 0.$$

Find the tangent line at 1, and verify that $(1, 2)$ is an inflection point.

12. Let

$$f(x) = \begin{cases} |x|, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Determine the sign of f' on each side of 0, and classify 0 as a local maximum, local minimum, or neither by comparing function values. Explain why the first-derivative test does not apply.

13. Let f be twice differentiable on an open interval I , with $f''(x) > 0$ throughout I . Prove that f' has at most one zero. If $f'(c) = 0$, prove that $f(c) < f(x)$ for every $x \in I \setminus \{c\}$. Does $f'' > 0$ guarantee that such a c exists? Give a reason or a counterexample.

14. Let f be twice differentiable on an open interval I , with $f'' > 0$ on I . For $a < b$ in I , prove

$$f\left(\frac{a+b}{2}\right) < \frac{f(a) + f(b)}{2}.$$

Hint: apply the Mean Value Theorem on the two halves of $[a, b]$, and compare the two derivative values.

15. Suppose f is continuous on $[1, 5]$, differentiable on $(1, 5)$, and

$$f(1) = 2, \quad -1 \leq f'(x) \leq 3 \quad (1 < x < 5).$$

Find the best lower and upper bounds for $f(5)$ that follow from these assumptions. Show that each bound is attained by a function satisfying all the assumptions.

16. Use the Mean Value Theorem to prove

$$|\sin x - \sin y| \leq |x - y| \quad (x, y \in \mathbb{R}).$$

Deduce an interval with rational endpoints containing $\sin(\pi/6 + 1/20)$, without using decimal approximations.

17. Let f be differentiable on an open interval I , and suppose $f'(x) \geq m > 0$ for all $x \in I$. Prove that f is one-to-one. If g denotes its inverse on $f(I)$, prove

$$|g(u) - g(v)| \leq \frac{1}{m}|u - v| \quad (u, v \in f(I)).$$

Do not assume a formula for the derivative of g .

18. Prove that the equation

$$\ln x + x = 2$$

has exactly one positive solution, and that this solution lies in $(3/2, 2)$.

19. Prove that $x^3 - 3x + 1 = 0$ has exactly three real solutions: one in each of $(-2, -1)$, $(0, 1)$, and $(1, 2)$. Use the derivative to justify that there are no other real solutions.
20. Suppose f is twice differentiable on an open interval containing c , and $(c, f(c))$ is an inflection point. Prove that $f''(c) = 0$. *Hint:* consider the monotonicity of f' on the two sides of c , and apply Fermat's theorem to f' .