

Homework 10: Solutions

Give exact answers and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. Local extrema are considered only at interior points of the stated domain. Use the strict tangent-line definition of concavity from Lecture 11.

1. The critical numbers are $-2, 1, 3$. The sign table is

x	$(-\infty, -2)$	$(-2, 1)$	$(1, 3)$	$(3, \infty)$
f'	+	-	-	+

Thus f is strictly increasing on $(-\infty, -2)$ and $(3, \infty)$, and strictly decreasing on $(-2, 3)$. To join the two decreasing pieces across 1, use continuity there: values to its left are greater than $f(1)$, and values to its right are smaller. There is a strict local maximum at -2 , a strict local minimum at 3 , and no extremum at 1 . The values of these extrema are not specified by the derivative alone.

2. Writing out the absolute value gives

$$f(x) = \begin{cases} x^2 - 4, & |x| \geq 2, \\ 4 - x^2, & |x| < 2. \end{cases}$$

Away from ± 2 , the derivative is $2x$ on the outer intervals and $-2x$ on $(-2, 2)$. At -2 the one-sided derivatives are -4 and 4 ; at 2 they are -4 and 4 . The critical numbers are therefore $[-2, 0, 2]$. The derivative signs on the four consecutive intervals are $-, +, -, +$. Continuity and the first-derivative test give strict local minima $f(-2) = f(2) = 0$ and a strict local maximum $f(0) = 4$.

3. The quotient rule gives

$$f'(x) = \frac{1 - \ln x}{x^2}, \quad f''(x) = \frac{2 \ln x - 3}{x^3}.$$

Both derivatives exist throughout the domain. The only critical number is e . Since $f''(e) = -1/e^3 < 0$, the second-derivative test gives a strict local maximum $f(e) = 1/e$.

4. We have $f'(x) = \cos x - 1$ and $f''(x) = -\sin x$. Thus $f'(0) = f''(0) = 0$, so the second-derivative test is inconclusive. For $0 < |x| < \pi$, $\cos x < 1$, so $f'(x) < 0$ on both sides of 0. The first-derivative test shows that 0 is neither a local maximum nor a local minimum.
5. Since $f(0) = 0$, the difference quotient is h for $h < 0$ and $-h$ for $h > 0$; both tend to zero. Hence $f'(0) = 0$. For $x < 0$, $f'(x) = 2x$, and for $x > 0$, $f'(x) = -2x$. Therefore

$$\frac{f'(h) - f'(0)}{h} = \begin{cases} 2, & h < 0, \\ -2, & h > 0, \end{cases}$$

so $f''(0)$ does not exist. The function is continuous at 0, concave upward on $(-\infty, 0)$, and concave downward on $(0, \infty)$. Thus $(0, 0)$ is an inflection point. There is no local extremum: values immediately to the left are positive, and values immediately to the right are negative.

6. The domain is $\mathbb{R}$, and

$$f'(x) = \frac{2x}{1+x^2}, \quad f''(x) = \frac{2(1-x^2)}{(1+x^2)^2}.$$

The denominator is positive. Thus the graph is concave upward on $(-1, 1)$ and concave downward on $(-\infty, -1)$ and $(1, \infty)$. Continuity and the changes of concavity give exactly two inflection points: $(-1, \ln 2)$ and $(1, \ln 2)$.

7. The zeros of f' are $-2, 0, 2$. Its signs give

Behavior of f	Intervals
increasing	$(-2, 0), (2, 3)$
decreasing	$(-3, -2), (0, 2)$

The sign changes give local minima at $x = -2, 2$ and a local maximum at $x = 0$. The graph of f' is strictly increasing on $(-3, -1)$ and $(1, 3)$, and strictly decreasing on $(-1, 1)$. Thus f is concave upward on the first two intervals and downward on the third. Differentiability makes f continuous, so the concavity changes at $x = -1, 1$ give inflection points. Although f'' does not exist at those two corners of the graph of f' , this is not required for an inflection point of f .

8. (a) For $x > 0$, $(\sqrt{x})'' = -1/(4x^{3/2}) < 0$, so the graph is concave downward. Its tangent at 1 is $y = 1 + (x - 1)/2 = (x + 1)/2$. The strict tangent inequality gives $\sqrt{x} < (x + 1)/2$ for $x \neq 1$; at $x = 1$ there is equality.
- (b) Apply (a) with $x = u/v > 0$ and multiply by $v > 0$:

$$\sqrt{uv} \leq \frac{u + v}{2}.$$

Equality holds exactly when $u/v = 1$, that is, $u = v$.

9. The chain rule gives

$$g'(x) = 3f'(2 - x), \quad g''(x) = -3f''(2 - x).$$

Thus $g' > 0$ and $g'' < 0$ on $\mathbb{R}$: the function is strictly increasing and concave downward.

- 10.** For $a > 0$, $f_a''(x) = 12x^2 + 2a > 0$ everywhere, so the graph is concave upward. For $a = 0$, the derivative $4x^3$ is strictly increasing on $\mathbb{R}$, which gives the same conclusion even though $f_0''(0) = 0$. If $a < 0$, the tangent at 0 is $y = 0$, but for $0 < |x| < \sqrt{-a}$,

$$f_a(x) = x^2(x^2 + a) < 0.$$

This contradicts the defining tangent inequality for concavity upward on $\mathbb{R}$. The answer is $\boxed{a \geq 0}$.

- 11.** Since $f''(1) = 6 + 2a = 0$, we obtain $a = -3$. The condition $f'(1) = 3 + 2a + b = 3$ then gives $b = 6$, and $f(1) = 1 + a + b + c = 2$ gives $c = -2$. Thus $\boxed{a = -3, b = 6, c = -2}$. The tangent is $y = 2 + 3(x - 1)$, or $\boxed{y = 3x - 1}$. For this polynomial, $f''(x) = 6(x - 1)$, which changes from negative to positive at 1. Continuity therefore makes $\boxed{(1, 2)}$ an inflection point.

- 12.** For $x < 0$, $f'(x) = -1$; for $x > 0$, $f'(x) = 1$. Nevertheless, when $0 < |x| < 1$,

$$f(x) = |x| < 1 = f(0).$$

Thus $\boxed{0 \text{ is a strict local maximum}}$. The function is not continuous at 0, because its limit there is 0 whereas its value is 1. The first-derivative test requires continuity at the point being classified, so it cannot be applied.

- 13.** Since $f'' > 0$, the derivative f' is strictly increasing on I ; hence it has at most one zero. Also f is concave upward, so if $f'(c) = 0$, its strict tangent inequality gives

$$f(x) > f(c) + f'(c)(x - c) = f(c) \quad (x \neq c).$$

Thus c is the unique point of absolute minimum on I . Existence is not guaranteed: $f(x) = e^x$ on $\mathbb{R}$ has $f''(x) > 0$ everywhere but $f'(x) = e^x$ never vanishes.

- 14.** Put $m = (a + b)/2$ and $d = (b - a)/2 > 0$. MVT supplies $u \in (a, m)$ and $v \in (m, b)$ such that

$$f(m) - f(a) = f'(u)d, \quad f(b) - f(m) = f'(v)d.$$

Because $f'' > 0$, the derivative f' is strictly increasing. As $u < v$, it follows that $f'(u) < f'(v)$. Multiplying by $d > 0$ gives $f(m) - f(a) < f(b) - f(m)$, so

$$\boxed{f(m) < \frac{f(a) + f(b)}{2}}.$$

The MVT hypotheses hold because differentiability on I implies continuity on both closed subintervals.

- 15.** The derivative bounds give

$$-4 \leq f(5) - f(1) \leq 12, \quad \text{hence} \quad \boxed{-2 \leq f(5) \leq 14}.$$

The function $f(x) = 3 - x$ satisfies $f(1) = 2$, has derivative -1 , and attains $f(5) = -2$. The function $f(x) = 3x - 1$ satisfies $f(1) = 2$, has derivative 3 , and attains $f(5) = 14$. These examples show that neither bound can be improved.

- 16.** If $x = y$, both sides are zero. Otherwise MVT on the interval with endpoints x, y gives a point c between them with $\sin x - \sin y = \cos c(x - y)$. Since $|\cos c| \leq 1$,

$$|\sin x - \sin y| \leq |x - y|.$$

Take $x = \pi/6 + 1/20$ and $y = \pi/6$, where $\sin y = 1/2$. Then the desired value differs from $1/2$ by at most $1/20$, so it belongs to $[9/20, 11/20]$.

- 17.** For $x < y$ in I , MVT gives

$$f(y) - f(x) = f'(c)(y - x) \geq m(y - x) > 0$$

for some $c \in (x, y)$. Thus f is strictly increasing and one-to-one, so its inverse on $f(I)$ exists. Interchanging the points when needed gives

$$|f(y) - f(x)| \geq m|y - x| \quad (x, y \in I).$$

Now take $x = g(u)$, $y = g(v)$. Since $f(g(u)) = u$ and $f(g(v)) = v$,

$$|g(u) - g(v)| \leq \frac{|u - v|}{m}.$$

The case $u = v$ is immediate. No derivative of the inverse was used.

- 18.** Let $F(x) = \ln x + x - 2$ on $(0, \infty)$. It is continuous and $F'(x) = 1/x + 1 > 0$, so it is strictly increasing and has at most one zero. The logarithmic inequality from class, strict for $t > 0$, gives $\ln(3/2) < 1/2$. Consequently,

$$F(3/2) = \ln(3/2) - 1/2 < 0, \quad F(2) = \ln 2 > 0.$$

IVT gives a zero in $(3/2, 2)$, and strict increase makes it the unique positive solution.

19. For $p(x) = x^3 - 3x + 1$,

$$p(-2) = -1, \quad p(-1) = 3, \quad p(0) = 1, \quad p(1) = -1, \quad p(2) = 3.$$

Continuity and IVT give at least one zero in each of the three specified, disjoint intervals. Moreover,

$$p'(x) = 3(x-1)(x+1).$$

The polynomial is strictly increasing on $(-\infty, -1)$, strictly decreasing on $(-1, 1)$, and strictly increasing on $(1, \infty)$. Each of these intervals contains at most one zero; each already contains one of the three zeros obtained above. The remaining points $-1, 1$ are not zeros. Hence there are exactly three.

20. Suppose first that the graph is concave upward immediately to the left of c and concave downward immediately to the right. The tangent characterization shows that f' is strictly increasing on the left interval and strictly decreasing on the right. Since $f''(c)$ exists, f' is differentiable, hence continuous, at c . For a fixed nearby $x < c$, let $y \rightarrow c^-$ with $x < y < c$. The inequality $f'(x) < f'(y)$ yields $f'(x) \leq f'(c)$. Similarly, for nearby $x > c$, continuity from the right and decrease give $f'(x) \leq f'(c)$. Thus f' has a local maximum at c . Fermat's theorem applied to f' gives

$$f''(c) = (f')'(c) = 0.$$

If the concavity changes in the opposite direction, f' has a local minimum, and Fermat gives the same conclusion. This proves a necessary condition under twice differentiability, not its converse.