

Homework 9

For Thursday, October 15, 2026.

Give exact answers and justify your reasoning. Do not use electronic tools. All functions are real-valued, and angles are measured in radians. Use the material covered through Section 5 of Lecture 10. For local extrema, consider only interior points of the stated domain.

1. Let

$$f(x) = \frac{(x-1)^2}{x+1}.$$

Find the domain and all critical numbers of f . Explain why -1 is not a critical number.

2. Find the absolute maximum and minimum of

$$f(x) = x^4 - 4x^2 \quad \text{on } [-2, 1].$$

Give all points where they are attained, and show the full list of values used in the closed-interval method.

3. Let $g(x) = \sqrt{|x-1|}$ on $[-3, 2]$. Find all interior critical numbers and the absolute extrema, including where they are attained. Explain why Fermat's theorem does not require the derivative to be zero at the minimum.

4. Find real numbers a, b such that

$$f(x) = x^3 + ax^2 + bx$$

satisfies $f(1) = 2$ and has a local maximum at 1. Use Fermat's theorem to find the possible coefficients, then verify the local maximum directly by examining $f(1+h) - f(1)$ for small nonzero h .

5. Check all hypotheses of Rolle's theorem for $f(x) = \sin x$ on $[0, \pi]$, and find every $c \in (0, \pi)$ satisfying its conclusion.

6. For $f(x) = 1/x^2$ on its natural domain, we have $f(-1) = f(1)$. Can Rolle's theorem be applied on $[-1, 1]$? Does $f'(x) = 0$ at any point of $(-1, 1)$ where the derivative is defined? Explain both answers.

7. Choose $\alpha, \beta, \gamma \in \mathbb{R}$ so that

$$f(x) = \begin{cases} x^2 + \alpha x, & 0 \leq x \leq 1, \\ \beta x + \gamma, & 1 < x \leq 2 \end{cases}$$

satisfies all the hypotheses of Rolle's theorem on $[0, 2]$. For your choices, find every point supplied by its conclusion.

8. Suppose f is continuous on $[-2, 5]$, differentiable on $(-2, 5)$, and

$$f(-2) = f(1) = f(5) = 0.$$

Prove that f' has at least two distinct zeros in $(-2, 5)$.

9. Apply the Mean Value Theorem to $f(x) = 1/x$ on $[1, 4]$. Find every point c supplied by the theorem, and find the equation of the tangent line at each such point. Check that its slope equals the slope of the secant joining the endpoints.
10. For $f(x) = x^3$ on $[-1, 1]$, find all points supplied by the Mean Value Theorem. Does this example have a unique such point? Is the midpoint one of them?

11. Let $f(x) = \sqrt[3]{x}$ on $[-1, 1]$.

- (a) Does f satisfy the hypotheses of the Mean Value Theorem?
 (b) Nevertheless, find every $c \in (-1, 1)$ for which

$$f'(c) = \frac{f(1) - f(-1)}{2}.$$

12. Suppose f is continuous on $[0, 3]$, differentiable on $(0, 3)$, and

x	0	1	3
$f(x)$	0	2	1

Prove each statement:

- (a) Some $u \in (0, 1)$ satisfies $f'(u) = 2$.
 (b) Some $v \in (1, 3)$ satisfies $f'(v) = -1/2$.
 (c) Some $w \in (0, 3)$ satisfies $f'(w) = 0$.

You are not given that f' is continuous. For (c), use the Extreme Value Theorem and Fermat's theorem.

13. A differentiable function F on $\mathbb{R}$ satisfies

$$F'(x) = \frac{2x}{1+x^2}, \quad F(0) = 3.$$

Determine $F(x)$ for every real x . *Hint:* compare its derivative with that of $\ln(1+x^2)$.

14. For $x \neq 0$, let

$$H(x) = \arctan x + \arctan(1/x).$$

Show that $H'(x) = 0$ throughout its domain. Determine $H(x)$ on each of the intervals $(-\infty, 0)$ and $(0, \infty)$, and explain why the constants need not agree.

15. Suppose F is differentiable on $\mathbb{R}$, with $F'(x) = F(x)$ for every x and $F(0) = 2$. Prove that $F(x) = 2e^x$ for every real x . *Hint:* differentiate $e^{-x}F(x)$.

16. Find the maximal open intervals on which

$$f(x) = \frac{x}{1+x^2}$$

is strictly increasing or strictly decreasing. Justify your answers using the derivative and the monotonicity criterion.

17. For a real parameter a , let $f_a(x) = x^3 + ax$ on $\mathbb{R}$.

- (a) Find all a for which f_a is nondecreasing on $\mathbb{R}$.
- (b) Among those values of a , which make f_a strictly increasing?

For excluded values in (a), exhibit two points that violate the definition of nondecreasing.

18. Let f be nondecreasing on an open interval I , and suppose it is differentiable at $c \in I$. Prove directly from the derivative definition that $f'(c) \geq 0$. Explain the sign of the difference quotient for both positive and negative h .

19. Let f be differentiable on $\mathbb{R}$, with

$$f'(x) > 0 \ (x < -1), \quad f'(x) = 0 \ (-1 \leq x \leq 1), \quad f'(x) < 0 \ (x > 1).$$

- (a) Prove that f is constant on $[-1, 1]$.
- (b) Compare $f(-2)$ with $f(0)$, and $f(2)$ with $f(0)$.
- (c) Do the assumptions determine which is larger, $f(-2)$ or $f(2)$? Justify your answer by giving two differentiable functions satisfying the stated derivative conditions and giving opposite strict orderings of these two values.

20. Suppose f is continuous on $[1, 3]$, differentiable on $(1, 3)$, and $f(1) = 2$, $f(3) = 6$. Prove that at some $c \in (1, 3)$, the tangent line to $y = f(x)$ passes through the origin. *Hint:* apply Rolle's theorem to $f(x)/x$.