

Homework 9: Solutions

1. The domain is $\mathbb{R} \setminus \{-1\}$. On that domain,

$$f'(x) = \frac{2(x-1)(x+1) - (x-1)^2}{(x+1)^2} = \frac{(x-1)(x+3)}{(x+1)^2}.$$

Thus the critical numbers are $\boxed{-3, 1}$. The derivative exists at every point of the domain. Although its formula is undefined at -1 , that point is outside the domain of f , so it is not a critical number.

2. Polynomial continuity gives the EVT hypotheses. Since

$$f'(x) = 4x(x^2 - 2),$$

the interior critical numbers in $(-2, 1)$ are $-\sqrt{2}$ and 0 . The other zero, $\sqrt{2}$, is outside the interval. The complete comparison is

x	-2	$-\sqrt{2}$	0	1
$f(x)$	0	-4	0	-3

The absolute maximum is $\boxed{0}$, attained at $\boxed{-2, 0}$. The absolute minimum is $\boxed{-4}$, attained only at $\boxed{-\sqrt{2}}$.

3. The function is continuous on $[-3, 2]$. For interior points other than 1 ,

$$g'(x) = \begin{cases} -1/(2\sqrt{1-x}), & x < 1, \\ 1/(2\sqrt{x-1}), & x > 1. \end{cases}$$

These values never vanish. At 1 , the difference quotient is $\sqrt{|h|}/h$, which has no finite limit. Thus the only interior critical number is $\boxed{1}$. The candidate values are

$$g(-3) = 2, \quad g(1) = 0, \quad g(2) = 1.$$

Hence the absolute maximum is $\boxed{2}$ at $\boxed{-3}$, and the absolute minimum is $\boxed{0}$ at $\boxed{1}$. Fermat's differentiability hypothesis fails at the minimum.

4. The value condition gives $1 + a + b = 2$, or $a + b = 1$. Fermat's theorem is applicable to this polynomial at an interior local maximum, so

$$f'(1) = 3 + 2a + b = 0.$$

Solving gives $\boxed{a = -4, b = 5}$. For these coefficients,

$$f(1+h) - f(1) = h^3 - h^2 = h^2(h-1).$$

If $0 < |h| < 1/2$, then $h^2 > 0$ and $h - 1 < 0$, so this difference is negative. Therefore 1 is indeed a strict local maximum, with value 2. The necessary condition from Fermat has been followed by a direct verification.

5. Sine is continuous on $[0, \pi]$, differentiable on $(0, \pi)$, and $\sin 0 = \sin \pi = 0$. All three hypotheses hold. Since $f'(x) = \cos x$, the unique point in $(0, \pi)$ with $f'(c) = 0$ is $\boxed{c = \pi/2}$.
6. Rolle's theorem cannot be applied: f is not defined at 0, and therefore is not continuous on $[-1, 1]$ or differentiable throughout $(-1, 1)$. For $x \neq 0$,

$$f'(x) = -\frac{2}{x^3},$$

which is never zero. Equal endpoint values alone do not give the conclusion of Rolle's theorem.

7. Continuity at 1, differentiability at 1, and equal endpoint values respectively require

$$1 + \alpha = \beta + \gamma, \quad 2 + \alpha = \beta, \quad 2\beta + \gamma = 0.$$

Solving gives

$$\alpha = -\frac{3}{2}, \quad \beta = \frac{1}{2}, \quad \gamma = -1.$$

The pieces are polynomials and their values and derivatives match at 1. Hence the function is continuous on $[0, 2]$ and differentiable on $(0, 2)$, with $f(0) = f(2) = 0$. For $0 < x < 1$, $f'(x) = 2x - 3/2$; for $1 \leq x < 2$, $f'(x) = 1/2$. Thus the only point with zero derivative is $c = 3/4$.

8. Apply Rolle's theorem to $[-2, 1]$. The restrictions inherit continuity and differentiability, and the endpoint values agree, so there is $c_1 \in (-2, 1)$ with $f'(c_1) = 0$. Applying Rolle to $[1, 5]$ similarly gives $c_2 \in (1, 5)$ with $f'(c_2) = 0$. The two open intervals are disjoint, so $c_1 \neq c_2$, as required.
9. The reciprocal function is continuous on $[1, 4]$ and differentiable on $(1, 4)$. Its secant slope is

$$\frac{1/4 - 1}{4 - 1} = -\frac{1}{4}.$$

The equation $-1/c^2 = -1/4$ gives $c = \pm 2$, but only $c = 2$ belongs to $(1, 4)$. Since $f(2) = 1/2$, the tangent line is

$$y - \frac{1}{2} = -\frac{1}{4}(x - 2), \quad \text{or} \quad y = 1 - \frac{x}{4}.$$

Its slope is exactly the secant slope.

10. The polynomial satisfies the continuity and differentiability hypotheses. Its secant slope is

$$\frac{1 - (-1)}{1 - (-1)} = 1.$$

Thus $3c^2 = 1$, giving $\boxed{c = \pm 1/\sqrt{3}}$. Both lie in $(-1, 1)$, so the point is not unique. The midpoint 0 is not one of them, because $f'(0) = 0 \neq 1$.

11. (a) The real cube-root function is continuous on $[-1, 1]$, but at 0 its difference quotient is

$$\frac{\sqrt[3]{h}}{h} = \frac{1}{(\sqrt[3]{h})^2},$$

which tends to $+\infty$. It is not differentiable at 0, so the hypotheses do not hold.

(b) For $x \neq 0$, differentiating $(f(x))^3 = x$ gives $f'(x) = 1/[3(\sqrt[3]{x})^2]$. The secant slope is 1, so

$$(\sqrt[3]{c})^2 = \frac{1}{3}, \quad \boxed{c = \pm \frac{1}{3\sqrt{3}}}.$$

Both points are in $(-1, 1)$. Thus this example has points satisfying the conclusion even though the theorem's hypotheses fail.

12. (a) Apply MVT on $[0, 1]$: some $u \in (0, 1)$ has $f'(u) = [f(1) - f(0)]/(1 - 0) = \boxed{2}$.

(b) Apply MVT on $[1, 3]$: some $v \in (1, 3)$ has $f'(v) = [f(3) - f(1)]/(3 - 1) = \boxed{-1/2}$.

(c) EVT gives an attained absolute maximum on $[0, 3]$. Its value is at least $f(1) = 2$, strictly greater than both endpoint values 0 and 1. A maximizing point w is therefore in $(0, 3)$. It is an interior local maximum where f is differentiable, so Fermat gives $\boxed{f'(w) = 0}$. This argument does not assume continuity of f' .

13. Set $G(x) = \ln(1+x^2)$, which is defined and differentiable on $\mathbb{R}$. Then $G'(x) = 2x/(1+x^2) = F'(x)$. The zero-derivative criterion applied to $F - G$ on the interval $\mathbb{R}$ gives $F(x) - G(x) = C$. At zero, $C = F(0) - G(0) = 3$. Therefore

$$\boxed{F(x) = \ln(1+x^2) + 3 \quad (x \in \mathbb{R}).}$$

14. For $x \neq 0$, the chain rule gives

$$H'(x) = \frac{1}{1+x^2} + \frac{-1/x^2}{1+1/x^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0.$$

Thus H is constant on each of the two intervals. Evaluating at 1 and -1 , respectively, gives

$$\boxed{H(x) = \begin{cases} -\pi/2, & x < 0, \\ \pi/2, & x > 0. \end{cases}}$$

The entire domain $\mathbb{R} \setminus \{0\}$ is not an interval, so the zero-derivative criterion does not require the two constants to agree.

15. Let $G(x) = e^{-x}F(x)$. The product rule and the hypothesis give

$$G'(x) = e^{-x}(F'(x) - F(x)) = 0 \quad (x \in \mathbb{R}).$$

Consequently G is constant on $\mathbb{R}$. Its value at zero is $G(0) = 2$, so $e^{-x}F(x) = 2$. Multiplying by e^x yields $\boxed{F(x) = 2e^x}$ for every real x .

16. The function is differentiable everywhere, with

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2}.$$

The denominator is positive. The derivative is positive for $-1 < x < 1$, zero at ± 1 , and negative for $x < -1$ or $x > 1$. Hence the maximal open intervals are

$$\boxed{\text{increasing on } (-1, 1), \quad \text{decreasing on } (-\infty, -1) \text{ and } (1, \infty).}$$

The monotonicity criterion gives the stated strict inequalities on each interval. Extending any of these open intervals past a boundary would include an interval where the derivative has the opposite sign, so none can be enlarged with the same monotonicity.

17. (a) If $a \geq 0$, then $f'_a(x) = 3x^2 + a \geq 0$ for every x , so f_a is nondecreasing on $\mathbb{R}$. If $a < 0$, choose $h = \sqrt{-a/2} > 0$. Then

$$f_a(h) - f_a(0) = h(h^2 + a) = \frac{ah}{2} < 0.$$

The points $0 < h$ therefore violate nondecrease. The answer is $\boxed{a \geq 0}$.

(b) For $a > 0$, the derivative is everywhere positive, so the function is strictly increasing. When $a = 0$, for $x < y$,

$$y^3 - x^3 = (y - x)(x^2 + xy + y^2) > 0.$$

Indeed, $x^2 + xy + y^2 = (x + y/2)^2 + 3y^2/4$, which can vanish only when $x = y = 0$, impossible for $x < y$. Thus $a = 0$ also gives strict increase, and the answer is again $\boxed{a \geq 0}$.

18. Choose $|h|$ small enough that $c + h \in I$. If $h > 0$, nondecrease gives $f(c + h) - f(c) \geq 0$, so

$$\frac{f(c + h) - f(c)}{h} \geq 0.$$

If $h < 0$, then $f(c + h) - f(c) \leq 0$; division by the negative h again makes the quotient nonnegative.

The derivative exists, so its limit is nonnegative: $\boxed{f'(c) \geq 0}$.

- 19. (a)** Differentiability implies continuity. The zero-derivative criterion on $[-1, 1]$ gives constancy there; write its value as $C = f(0)$.
- (b)** MVT on $[-2, -1]$ gives $f(-1) - f(-2) = f'(\xi) > 0$ for some $\xi \in (-2, -1)$. Since $f(-1) = C$, we have $\boxed{f(-2) < f(0)}$. Similarly, MVT on $[1, 2]$ gives $f(2) - f(1) = f'(\eta) < 0$, so $\boxed{f(2) < f(0)}$.
- (c)** For any $A, B > 0$, define

$$f_{A,B}(x) = \begin{cases} -A(x+1)^2, & x < -1, \\ 0, & -1 \leq x \leq 1, \\ -B(x-1)^2, & x > 1. \end{cases}$$

The values match at the joins, and the one-sided derivatives there are all zero, so this function is differentiable on $\mathbb{R}$. Its derivatives on the outer intervals are $-2A(x+1) > 0$ on $(-\infty, -1)$ and $-2B(x-1) < 0$ on $(1, \infty)$. The derivative is zero on $[-1, 1]$, including its endpoints. Thus it satisfies every stated condition, and

$$f_{A,B}(-2) = -A, \quad f_{A,B}(2) = -B.$$

Taking $(A, B) = (1, 2)$ gives $f(-2) > f(2)$, whereas $(A, B) = (2, 1)$ gives $f(-2) < f(2)$. The ordering is not determined.

- 20.** Let $g(x) = f(x)/x$. The denominator is nonzero on $[1, 3]$, so g is continuous there and differentiable on $(1, 3)$. Moreover, $g(1) = 2 = g(3)$. Rolle's theorem therefore gives $c \in (1, 3)$ with

$$0 = g'(c) = \frac{cf'(c) - f(c)}{c^2}.$$

It follows that $f(c) = cf'(c)$. The tangent line at c is

$$y = f(c) + f'(c)(x - c) = f'(c)x,$$

which passes through $(0, 0)$, as required.