

Homework 8: Solutions

1. The fraction is defined when $x \neq -1$ and nonzero when $x \neq 1$. Thus the domain is $\mathbb{R} \setminus \{-1, 1\}$. On this domain,

$$F(x) = \ln|x-1| - \ln|x+1|, \quad F'(x) = \frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}.$$

The derivative has the same domain as F .

2. At $x = -2/3$, the value is $\operatorname{arcsec}(-2) = \arccos(-1/2) = 2\pi/3$. Since $|3x| > 1$ near this point,

$$y' = \frac{3}{|3x|\sqrt{9x^2-1}}, \quad y'(-2/3) = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

The tangent is

$$y - \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \left(x + \frac{2}{3} \right).$$

The absolute value is essential at this negative argument.

3. In (a), only the base varies; the real-power rule gives

$$\frac{d}{dx} x^{\sqrt{2}} = \sqrt{2} x^{\sqrt{2}-1}.$$

In (b), only the exponent varies; the exponential rule gives

$$\frac{d}{dx} (\sqrt{2})^x = (\sqrt{2})^x \ln \sqrt{2}.$$

In (c), both vary. Write $x^x = e^{x \ln x}$, or differentiate $\ln(x^x) = x \ln x$, to obtain

$$\frac{d}{dx} x^x = x^x (\ln x + 1).$$

All formulas apply on the stated interval $x > 0$.

4. For $x > -1$, $x \neq 0$,

$$\ln |f(x)| = \ln |x| + \sqrt{2} \ln(x+1), \quad \frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{\sqrt{2}}{x+1}.$$

Hence

$$\boxed{f'(x) = (x+1)^{\sqrt{2}} + \sqrt{2}x(x+1)^{\sqrt{2}-1}.$$

At zero, $f(0) = 0$, and

$$f'(0) = \lim_{h \rightarrow 0} \frac{h(1+h)^{\sqrt{2}}}{h} = \lim_{h \rightarrow 0} (1+h)^{\sqrt{2}} = \boxed{1}.$$

Continuity of the real power near 1 justifies the last limit. The logarithm is undefined at $f(0) = 0$, so the logarithmic calculation does not itself cover zero. The simplified derivative formula extends there with the value just proved.

5. For nonzero x close to zero, factor the quotient as

$$\frac{\arccos x - \arccos 0}{x} \frac{x}{\sin x}.$$

The first factor tends to $(\arccos)'(0) = -1$, and the second tends to 1, because $\sin x/x \rightarrow 1$. Thus the limit is $\boxed{-1}$. No rule for indeterminate quotients is needed.

6. Differentiability makes u continuous at zero. Since $u(0) = 2$, we have $u(x) > 0$ on a sufficiently small interval about zero. There $Y = e^{v \ln u}$ is defined and differentiable, with

$$Y' = u^v \left(v' \ln u + v \frac{u'}{u} \right).$$

Consequently

$$\boxed{Y'(0) = 2^3 \left(2 \ln 2 - \frac{3}{2} \right) = 16 \ln 2 - 12.}$$

7. For $x > 0$, $f'(x) = 1/(3x^{2/3})$. Thus $f(8) = 2$ and $f'(8) = 1/12$, so

$$L(x) = 2 + \frac{x - 8}{12}.$$

Since $8.12 - 8 = 3/25$, the requested estimate is

$$\sqrt[3]{8.12} \approx 2 + \frac{3/25}{12} = \frac{201}{100} = 2.01.$$

This is an approximation; no numerical error bound is asserted.

8. The given line gives $f(2) = 5$ and $f'(2) = -3$. Continuity makes f positive near 2, so $\ln f$ is defined there. Its value and derivative at 2 are

$$g(2) = \ln 5, \quad g'(2) = \frac{f'(2)}{f(2)} = -\frac{3}{5}.$$

Hence

$$L_g(x) = \ln 5 - \frac{3}{5}(x - 2).$$

9. For $h \neq 0$, write

$$f(1+h) - 4 = h \frac{f(1+h) - 4}{h}.$$

The right side tends to 0, so $f(1+h) \rightarrow 4$. Continuity at 1 then gives $f(1) = 4$. The given limit is therefore exactly the derivative definition, and $f'(1) = -2$. The linearization is $L(x) = 4 - 2(x - 1)$.

10. We have $f(0) = 2$, and for $h \neq 0$,

$$\frac{f(h) - f(0)}{h} = -1 + \frac{|h|^{3/2}}{h}, \quad \left| \frac{|h|^{3/2}}{h} \right| = \sqrt{|h|} \rightarrow 0.$$

Thus $f'(0) = -1$ and $L(x) = 2 - x$. The exact error is $|f(x) - L(x)| = |x|^{3/2}$. Given $\varepsilon > 0$, choose $\delta = \varepsilon^2$. Then $0 < |x| < \delta$ implies

$$|f(x) - L(x)| = |x|\sqrt{|x|} < \varepsilon|x|.$$

11. Since $f'(2) = -1/4$,

$$\boxed{dy = -\frac{h}{4}}, \quad \boxed{\Delta y = \frac{1}{2+h} - \frac{1}{2} = -\frac{h}{2(2+h)}}.$$

Subtracting gives

$$\boxed{\Delta y - dy = \frac{h^2}{4(2+h)}}.$$

For $0 < |h| < 1$, the numerator and denominator are positive. Therefore $\boxed{\Delta y > dy}$, for either sign of h .

12. The inverse-function derivative theorem applies at 1, because $f'(1) = 4 \neq 0$. It gives $g(2) = 1$ and $g'(2) = 1/4$. Consequently

$$\boxed{L_g(y) = 1 + \frac{y-2}{4}}, \quad \boxed{g(2.04) \approx 1 + \frac{1/25}{4} = \frac{101}{100} = 1.01}.$$

The stated domain assumption permits this estimate. Differentiability alone does not provide a numerical error bound at 2.04.

13. Suppose the displayed limit were zero. Multiplying by $|h|$ shows $|h| - A - Bh \rightarrow 0$. The left side also tends to $-A$, so $A = 0$. For positive h , the quotient then equals $1 - B$, which forces $B = 1$. For negative h , it equals $1 + B$, which forces $B = -1$. These are incompatible, so no such A, B exist.
14. Write the two expressions as $A + h(B + \rho(h))$ and $C + h(D + \sigma(h))$. Multiplication gives

$$\begin{aligned} f(a+h)g(a+h) &= AC + (AD + BC)h \\ &\quad + h[A\sigma(h) + C\rho(h) + h(B + \rho(h))(D + \sigma(h))]. \end{aligned}$$

Thus one choice is

$$\boxed{\tau(h) = A\sigma(h) + C\rho(h) + h(B + \rho(h))(D + \sigma(h))}.$$

The first two terms tend to zero. The last tends to $0 \cdot B \cdot D = 0$ by the limit laws, so $\tau(h) \rightarrow 0$. The given identities at $h = 0$ give $f(a) = A$, $g(a) = C$. The converse remainder characterization shows

$$\boxed{L_{fg}(x) = AC + (AD + BC)(x - a)}.$$

15. On $[-1, 2]$, we have $0 \leq (x-1)^2 \leq 4$, so $0 \leq f(x) \leq 4$. Equality at the upper end requires $x = 1$; equality at the lower end requires $x = -1$ (the other solution, $x = 3$, is outside the domain). Thus the maximum is $\boxed{4 \text{ at } 1}$, and the minimum is $\boxed{0 \text{ at } -1}$.

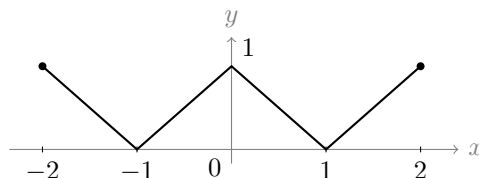
On $(-1, 2)$, the maximum is still $\boxed{4 \text{ at } 1}$. Every function value is positive. For any proposed minimum value $m > 0$, choose $0 < t < \min\{1, m/4\}$. Then $-1 + t \in (-1, 2)$, and

$$f(-1 + t) = 4t - t^2 < 4t < m.$$

This contradicts the defining property of a minimum. Hence $\boxed{\text{no absolute minimum}}$ exists on the open interval.

16. The graph joins $(-2, 1), (-1, 0), (0, 1), (1, 0), (2, 1)$ by straight segments:

$$f(x) = \begin{cases} -x - 1, & -2 \leq x \leq -1, \\ x + 1, & -1 \leq x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ x - 1, & 1 \leq x \leq 2. \end{cases}$$



The local minima are $\boxed{0 \text{ at } x = \pm 1}$; the only local maximum is $\boxed{1 \text{ at } x = 0}$. The absolute minimum is $\boxed{0 \text{ at } x = \pm 1}$, and the absolute maximum is $\boxed{1 \text{ at } x = -2, 0, 2}$. On the interior of each segment the slope is nonzero, so nearby values occur on both sides of the value at the point; no further local extrema occur. Endpoints are not local extrema under the stated convention.

17. (a) **Yes.** Absolute value is continuous on the closed bounded interval $[-1, 1]$; differentiability is not required.
- (b) **No guarantee.** The function is continuous on its stated domain, but $(0, 1]$ is not a closed interval.
- (c) **No guarantee.** The denominator is always positive, so k is continuous on $\mathbb{R}$, but $\mathbb{R}$ is unbounded. Failure of a hypothesis means the theorem does not give the conclusion; it is not, by itself, proof that an extremum is absent.
18. (a) The left limit at 1 is 1, so continuity at that endpoint holds exactly when $\boxed{c = 1}$. The function is continuous at all other points of $[0, 1]$ for every c .
- (b) The values from $[0, 1)$ are exactly $[0, 1)$. If $c \geq 1$, the absolute maximum is $\boxed{c \text{ at } x = 1}$. If $c < 1$, choose any r with $\max\{0, c\} < r < 1$. Every point in $[0, 1)$ also has a larger value at $(x + 1)/2$. Thus neither the endpoint value c nor any interior value can be a maximum. A maximum exists exactly when $\boxed{c \geq 1}$.
- (c) Comparing the extra value c with the minimum 0 on $[0, 1)$ gives

Parameter	Minimum value	Locations
$c < 0$	c	$x = 1$
$c = 0$	0	$x = 0, 1$
$c > 0$	0	$x = 0$

For example, when $c > 1$, both extrema exist even though the function is discontinuous at 1. Continuity is sufficient in EVT, not necessary for a particular function to attain extrema.

19. The negation is

$$\boxed{\text{For every } \delta > 0 \text{ there exists } x \in I \text{ such that } |x - c| < \delta \text{ and } f(x) > f(c).}$$

The strict inequality already ensures $x \neq c$. For $f(x) = x$ on $\mathbb{R}$, given any c and any $\delta > 0$, choose $x = c + \delta/2$. Then $|x - c| = \delta/2 < \delta$, while $f(x) = c + \delta/2 > c = f(c)$. Thus the negation holds at every c .

20. (a) The function $|f|$ is continuous on $[a, b]$. By EVT, it attains a minimum at some $c \in [a, b]$. Set $m = |f(c)|$. The hypothesis $f(c) \neq 0$ implies $m > 0$, and

$$|f(x)| \geq m \quad \text{for every } x \in [a, b].$$

Therefore

$$\boxed{\left| \frac{1}{f(x)} \right| \leq \frac{1}{m} \quad \text{for every } x \in [a, b],}$$

which proves boundedness. The fact that the minimum is attained is what makes it strictly positive.

(b) **No.** On $(0, 1)$, take $f(x) = x$. This is continuous and never zero. For any proposed $m > 0$, the point $x = \min\{m/2, 1/2\}$ lies in $(0, 1)$ and has $0 < f(x) < m$. Thus no positive lower bound exists. Also $1/x$ is unbounded: for any $K > 0$, $x = 1/(K + 1)$ lies in $(0, 1)$ and $1/x = K + 1 > K$.