

Homework 7: Solutions

1. For example, take $u(x) = 3x$, $v(t) = \cos t$, and $w(s) = (2 + s)^4$. Then $F = w \circ v \circ u$, and

$$F'(x) = 4(2 + \cos(3x))^3(-\sin(3x))3 = \boxed{-12(2 + \cos(3x))^3 \sin(3x)}.$$

Both the function and its derivative are defined on $\mathbb{R}$.

2. The outer derivatives are evaluated at different points:

$$(f \circ g)'(1) = f'(g(1))g'(1) = f'(2)(-1) = \boxed{-4},$$

$$(g \circ f)'(1) = g'(f(1))f'(1) = g'(3)2 = \boxed{10}.$$

3. Since $1 + e^{2x} > 0$, the quotient is differentiable for every real x . The quotient and chain rules give

$$F'(x) = \frac{2e^{2x}(1 + e^{2x}) - e^{2x}(2e^{2x})}{(1 + e^{2x})^2} = \boxed{\frac{2e^{2x}}{(1 + e^{2x})^2}}.$$

Put $z = e^{2x} > 0$. The equation $F'(x) = 1/2$ becomes

$$4z = (1 + z)^2 \iff (z - 1)^2 = 0.$$

Thus $e^{2x} = 1$, which holds precisely when $\boxed{x = 0}$. Substitution gives $F'(0) = 1/2$.

4. The logarithm requires $1 - x^2 > 0$, or $-1 < x < 1$. The chain rule gives

$$\boxed{F'(x) = -\frac{2x}{1 - x^2}, \quad -1 < x < 1.}$$

Both domains are $\boxed{(-1, 1)}$. The rational expression outside this interval is not a derivative of F , because F is not defined there.

5. The given tangent shows that $f(2) = 5$ and $f'(2) = -3$. For $H(x) = f(x^2 + 1)$, continuity of the inner function ensures that the composition is defined near 1. Then

$$H(1) = 5, \quad H'(1) = f'(2) \cdot 2 = -6.$$

The tangent is $\boxed{y - 5 = -6(x - 1)}$.

6. Let $u(h) = \ln(1 + 3h)$, which is defined on an open interval about zero and satisfies $u(0) = 0$. The limit is $u'(0)$. By the chain rule,

$$u'(h) = \frac{3}{1 + 3h}, \quad \boxed{u'(0) = 3}.$$

7. Differentiability at zero requires continuity there. Since $F(0) = 0$, this forces $a + b = 0$. Under that condition, the left difference quotient is

$$\frac{F(h) - F(0)}{h} = a \frac{e^{2h} - 1}{h} \rightarrow 2a \quad (h \rightarrow 0^-).$$

The right difference quotient $\ln(1+h)/h$ tends to 1. These limits follow from the known derivatives of e^{2x} and $\ln(1+x)$ at zero. Equality requires $2a = 1$, so

$$\boxed{a = \frac{1}{2}, \quad b = -\frac{1}{2}, \quad F'(0) = 1.}$$

These values also make the two one-sided difference quotients have the same finite limit, proving sufficiency.

8. Differentiate $f(f(x)) = x$ using the chain rule:

$$f'(f(x))f'(x) = 1.$$

Neither factor can be zero. If $f(a) = a$, this becomes $[f'(a)]^2 = 1$, so $\boxed{f'(a) = 1 \text{ or } -1}$. Both possibilities occur: $f(x) = x$ and $f(x) = -x$ satisfy the identity and fix zero.

9. Differentiating along a branch gives

$$e^y y' + y + xy' = e^x, \quad \boxed{y' = \frac{e^x - y}{e^y + x} \quad (e^y + x \neq 0)}.$$

At $(0, 0)$, the original equation is $1 = 1$, the denominator is 1, and the slope is 1. The tangent is $\boxed{y = x}$.

10. Differentiation yields

$$2x + y + (x + 2y)y' = 0, \quad y' = -\frac{2x + y}{x + 2y} \quad (x + 2y \neq 0).$$

Any horizontal tangent requires $2x + y = 0$, hence $y = -2x$. Substitution in the original equation gives $3x^2 = 3$. The only candidates are $(1, -2)$ and $(-1, 2)$. Their denominators $x + 2y$ are -3 and 3 , respectively, so both do give horizontal tangents. Thus the complete answer is

$$\boxed{(1, -2) : y = -2, \quad (-1, 2) : y = 2.}$$

11. For $x(y) = y^2$, we have $dx/dy = 2y$, equal to zero at $y = 0$. The tangent written with y as the independent variable is $x - 0 = 0(y - 0)$, or $\boxed{x = 0}$. For the two square-root branches, the right difference quotients are

$$\frac{\sqrt{h} - 0}{h} = \frac{1}{\sqrt{h}} \rightarrow +\infty, \quad \frac{-\sqrt{h} - 0}{h} = -\frac{1}{\sqrt{h}} \rightarrow -\infty \quad (h \rightarrow 0^+).$$

Neither gives a finite right derivative.

12. Factoring gives $(y - x^2)(y + x^2) = 0$, so the curve is the union of the graphs $y = x^2$ and $y = -x^2$. Their derivatives are $2x$ and $-2x$; both have tangent $\boxed{y = 0}$ at the origin. Implicit differentiation gives $2yy' = 4x^3$. At the origin this reduces to $0 = 0$, which supplies no slope. For $y \neq 0$, the formula $y' = 2x^3/y$ is valid, but its undefined value at the origin does not determine either tangent.

13. Since $f(1) = 3$ and f is one-to-one, $g(3) = 1$. Also $f'(t) = 3t^2 + 2$, so $f'(1) = 5 \neq 0$. The inverse formula gives

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

14. On $(-\pi/2, \pi/2)$, tangent is continuous and strictly increasing onto $\mathbb{R}$, and $(\tan t)' = \sec^2 t > 0$. For every $x \in \mathbb{R}$, its inverse value $y = \arctan x$ belongs to this open interval. The inverse-function derivative theorem applies and gives

$$\frac{d}{dx} \arctan x = \frac{1}{\sec^2(\arctan x)} = \boxed{\frac{1}{1+x^2}}, \quad x \in \mathbb{R},$$

using $\sec^2 y = 1 + \tan^2 y = 1 + x^2$. This denominator is positive for every real x ; every inverse value also lies where the original tangent derivative is nonzero.

15. The cubic is continuous and strictly increasing on $\mathbb{R}$. At $t = g(x) \neq 0$, its derivative $3t^2$ is nonzero. Therefore

$$g'(x) = \frac{1}{3(\sqrt[3]{x})^2}, \quad x \neq 0.$$

At zero, for $h \neq 0$,

$$\frac{g(h) - g(0)}{h} = \frac{\sqrt[3]{h}}{h} = \frac{1}{(\sqrt[3]{h})^2} \rightarrow +\infty.$$

Indeed, the positive denominator tends to zero. The limit is not finite, so $g'(0)$ does not exist under our definition. The failed hypothesis is $f'(g(0)) \neq 0$, since $f'(0) = 0$.

- 16.** The inverse sine takes its values in $[-\pi/2, \pi/2]$. On the middle interval it returns x . For the outer intervals, use $\sin(-\pi - x) = \sin x$ and $\sin(\pi - x) = \sin x$:

$$F(x) = \begin{cases} -\pi - x, & -\pi \leq x < -\pi/2, \\ x, & -\pi/2 \leq x \leq \pi/2, \\ \pi - x, & \pi/2 < x \leq \pi. \end{cases}$$

The displayed values all lie in the required inverse range. Differentiation on the open pieces gives

$$F'(x) = \begin{cases} -1, & -\pi < x < -\pi/2 \text{ or } \pi/2 < x < \pi, \\ 1, & -\pi/2 < x < \pi/2. \end{cases}$$

At $-\pi/2$, the one-sided slopes are -1 and 1 ; at $\pi/2$, they are 1 and -1 . Thus F is not differentiable at either of those two interior points.

- 17.** Fix any real a . For $h \neq 0$, the given identity and $f(0) = 1$ imply

$$\frac{f(a+h) - f(a)}{h} = \frac{f(a)f(h) - f(a)}{h} = f(a) \frac{f(h) - f(0)}{h}.$$

The last quotient tends to $f'(0)$, a finite number by hypothesis. The scalar-multiple limit law gives

$$f'(a) = f(a)f'(0).$$

This proves differentiability at a ; it was not assumed in forming the difference quotient. Since a was arbitrary, the conclusion holds on all of $\mathbb{R}$.

18. (a) For $h \neq 0$,

$$\frac{g(h) - g(0)}{h} = h \sin(1/h), \quad |h \sin(1/h)| \leq |h| \rightarrow 0.$$

Squeeze gives $\boxed{g'(0) = 0}$.

(b) The exponential is differentiable at $g(0) = 0$, and part (a) establishes differentiability of the inner function. Thus

$$\boxed{H'(0) = e^{g(0)} g'(0) = 1 \cdot 0 = 0.}$$

(c) For every positive integer n , take $h = 1/(n\pi)$. Then $h \neq 0$ and $g(h) = h^2 \sin(n\pi) = 0 = g(0)$. Such points lie arbitrarily close to zero. Dividing by $g(h) - g(0)$ would divide by zero at all these points. The chain-rule proof using the continuously extended quotient avoids this division and remains applicable.

19. One choice is $f(t) = t^2$ and $g(x) = |x|$. Both are nonconstant; f is differentiable everywhere and g is continuous everywhere. At zero,

$$\frac{g(h) - g(0)}{h} = \frac{|h|}{h} = \begin{cases} 1, & h > 0, \\ -1, & h < 0, \end{cases}$$

so g is not differentiable there. Nevertheless $(f \circ g)(x) = |x|^2 = x^2$, hence $\boxed{(f \circ g)'(0) = 0}$. The chain-rule hypotheses are sufficient, not necessary, for differentiability of a composition.

20. The two rays have slopes -1 and 1 , so $f(t) = 1 - t$ for $t \leq 1$, and $f(t) = t - 1$ for $t \geq 1$. Consequently

$$H(x) = \begin{cases} 1 - x^2, & |x| \leq 1, \\ x^2 - 1, & |x| \geq 1, \end{cases} \quad \boxed{H'(x) = \begin{cases} -2x, & |x| < 1, \\ 2x, & |x| > 1. \end{cases}}$$

The two formulas for H agree at $x = \pm 1$. At -1 , the left and right slopes are -2 and 2 ; at 1 , they are -2 and 2 . Thus H fails to be differentiable exactly at $\boxed{x = -1, 1}$. It is differentiable at zero, with derivative zero.