

Homework 6: Solutions

1. Fix $a \neq 0$. If $0 < |h| < |a|/2$, then $a + h \neq 0$, and

$$\frac{f(a+h) - f(a)}{h} = \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = -\frac{1}{a(a+h)}.$$

As $h \rightarrow 0$, the denominator tends to $a^2 \neq 0$. The quotient limit law gives

$$f'(a) = -\frac{1}{a^2} \quad (a \neq 0).$$

Thus the derivative has domain $\mathbb{R} \setminus \{0\}$. No value of the original function is defined at zero.

2. Write $-2/x^3 = -2x^{-3}$. The power and linearity rules give

$$p'(x) = 12x^3 - 10x + \frac{6}{x^4}, \quad x \neq 0.$$

Both p and its derivative have domain $\mathbb{R} \setminus \{0\}$.

3. By the product rule,

$$(fg)'(1) = f'(1)g(1) + f(1)g'(1) = (-3)4 + 2(5) = \boxed{-2}.$$

Since $g(1) = 4 \neq 0$, the quotient rule applies:

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \frac{-12 - 10}{16} = \boxed{-\frac{11}{8}}.$$

4. The original domain is $\mathbb{R} \setminus \{2\}$, and on it

$$r(x) = \frac{(x-2)(x+2)}{x-2} = x+2.$$

Consequently $r'(x) = 1$, with domain $\mathbb{R} \setminus \{2\}$. Cancellation does not insert the missing point into either domain.

If R is differentiable at 2, it is continuous there, so

$$c = R(2) = \lim_{x \rightarrow 2} r(x) = 4.$$

Conversely, for $c = 4$ we have $R(x) = x + 2$ on all of $\mathbb{R}$; hence it is differentiable and $\boxed{R'(2) = 1}$.

Thus $\boxed{c = 4}$ is necessary and sufficient.

5. The denominator vanishes precisely when $x = (2k + 1)\pi$, $k \in \mathbb{Z}$. At every other real number,

$$\begin{aligned} F'(x) &= \frac{\cos x(1 + \cos x) - \sin x(-\sin x)}{(1 + \cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} = \boxed{\frac{1}{1 + \cos x}}. \end{aligned}$$

Both domains are $\boxed{\mathbb{R} \setminus \{(2k + 1)\pi : k \in \mathbb{Z}\}}$.

6. The curve meets the y -axis at $(0, 1)$. Since $(3^x)' = 3^x \ln 3$, the slope there is $\ln 3$. The tangent is

$$\boxed{y = 1 + x \ln 3}.$$

Setting $y = 0$ gives the x -intercept $\boxed{(-1/\ln 3, 0)}$. Here $\ln 3 > 0$, so the division is valid.

7. The product rule gives

$$\begin{aligned} h'(x) &= e^x(x^2 - 3x + 1) + e^x(2x - 3) \\ &= e^x(x^2 - x - 2) = e^x(x - 2)(x + 1). \end{aligned}$$

Since $e^x > 0$ for every real x , the derivative vanishes exactly at $x = -1$ and $x = 2$. The points and tangent lines are

$$\boxed{(-1, 5/e), \quad y = 5/e,} \quad \boxed{(2, -e^2), \quad y = -e^2}.$$

The function is differentiable everywhere, so these are all horizontal tangents.

8. Write $p(x) = Ax^3 + Bx^2 + Cx + D$. The conditions at zero give $D = 1$ and $C = 0$. Since $p'(x) = 3Ax^2 + 2Bx + C$, the conditions at one become

$$A + B = -1, \quad 3A + 2B = 0.$$

Subtracting twice the first equation from the second gives $A = 2$, and then $B = -3$. Thus

$$\boxed{p(x) = 2x^3 - 3x^2 + 1}.$$

Substitution verifies all four conditions. Every polynomial of degree at most three has the displayed form, and its coefficients were uniquely forced by the conditions; this also proves uniqueness.

9. Differentiability at 1 requires continuity there. The left-hand limit is $a + b$, whereas $f(1) = 2$; therefore $a + b = 2$. Under this condition, for $h < 0$,

$$\frac{f(1+h) - f(1)}{h} = \frac{a(1+h) + b - 2}{h} = a.$$

For $h > 0$,

$$\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 + 1 - 2}{h} = 2 + h.$$

The two one-sided limits agree precisely when $a = 2$; then $b = 0$. These values give a two-sided derivative equal to 2, so they are sufficient as well as necessary. Hence

$$a = 2, \quad b = 0, \quad f'(x) = \begin{cases} 2, & x < 1, \\ 2, & x = 1, \\ 2x, & x > 1. \end{cases}$$

10. For $x > 0$, $f(x) = x^2$, so $f'(x) = 2x$. For $x < 0$, $f(x) = -x^2$, so $f'(x) = -2x$. At zero the difference quotient is

$$\frac{f(h) - f(0)}{h} = \frac{h|h|}{h} = |h| \rightarrow 0.$$

Thus f is differentiable everywhere and $\boxed{f'(x) = 2|x|}$ for all $x \in \mathbb{R}$. Moreover $|f'(x) - f'(0)| = 2|x| \rightarrow 0$, so f' is continuous at zero.

11. The product fg is differentiable on I . Apply the two-function rule first to $(fg)h$, then to fg :

$$(fgh)' = (fg)'h + (fg)h' = \boxed{f'gh + fg'h + fgh'}.$$

All factors on the right are evaluated at the same $x \in I$. For $u(x) = xe^x \sin x$, this gives

$$\boxed{u'(x) = e^x \sin x + xe^x \sin x + xe^x \cos x, \quad x \in \mathbb{R}.}$$

12. Let $u(x) = e^x \cos x$, so $u(0) = 1$. The given limit is

$$\lim_{x \rightarrow 0} \frac{u(x) - u(0)}{x - 0} = u'(0).$$

By the product rule, $u'(x) = e^x \cos x - e^x \sin x$. Consequently the limit is $\boxed{u'(0) = 1}$.

13. Let $H = fg$. Both f and H are differentiable at a . Continuity of f , together with $f(a) \neq 0$, gives an $r > 0$ such that

$$|x - a| < r \implies |f(x) - f(a)| < \frac{|f(a)|}{2}, \quad |f(x)| > \frac{|f(a)|}{2} > 0.$$

Choose r small enough that this interval lies in the common domain. There $g = H/f$, so the quotient rule gives

$$g'(a) = \frac{H'(a)f(a) - H(a)f'(a)}{f(a)^2} = \boxed{\frac{(fg)'(a) - g(a)f'(a)}{f(a)}}.$$

This applies the quotient rule to two functions already known to be differentiable; it does not assume that g is differentiable.

For a counterexample, take $a = 0$, $f(x) = x$ and $g(x) = |x|$. Then f is differentiable, and the difference quotient of $fg = x|x|$ at zero is $|h| \rightarrow 0$, so fg is differentiable there. But g is not differentiable at zero, and $f(0) = 0$. Thus the nonzero hypothesis cannot be dropped.

14. Since $f(a) = 0$, we have $(fg)(a) = 0$. For nonzero h small enough that $a + h \in I$,

$$\left| \frac{(fg)(a+h) - (fg)(a)}{h} \right| = |g(a+h)| \left| \frac{f(a+h) - f(a)}{h} \right| \leq M \left| \frac{f(a+h) - f(a)}{h} \right|.$$

The last expression tends to $M|f'(a)| = 0$. Squeeze therefore gives the required difference-quotient limit 0. Equivalently, given $\varepsilon > 0$, the definition of $f'(a) = 0$ supplies a $\delta > 0$ with $(a-\delta, a+\delta) \subset I$ such that

$$0 < |h| < \delta \implies \left| \frac{f(a+h) - f(a)}{h} \right| < \frac{\varepsilon}{M}.$$

The displayed bound then makes the difference quotient for fg have absolute value less than ε . Thus $\boxed{(fg)'(a) = 0}$.

15. Fix $x \in \mathbb{R}$. Evenness gives $f(-x+h) = f(x-h)$ and $f(-x) = f(x)$. Therefore

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h}.$$

Put $t = -h$. This is a change of variable in a two-sided limit, so

$$f'(-x) = -\lim_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t} = \boxed{-f'(x)}.$$

Thus f' is odd. At zero, $f'(0) = -f'(0)$, so $\boxed{f'(0) = 0}$.