

Homework 5: Solutions

1. The statement means

$$\forall M > 0 \exists \delta > 0 \forall x \in \mathbb{R} \setminus \{2\} : 0 < |x - 2| < \delta \implies f(x) < -M.$$

Its logical negation is

$$\exists M_0 > 0 \forall \delta > 0 \exists x \in \mathbb{R} \setminus \{2\} : 0 < |x - 2| < \delta \text{ and } f(x) \geq -M_0.$$

Negating a limit of $-\infty$ does not assert a limit of $+\infty$.

2. The limit is $\boxed{1}$. By the triangle inequality, for $0 < |x| < 1$,

$$|f(x) - 1| \leq |f(x) - (1 + 2x)| + 2|x| \leq 3x^2 + 2|x| \leq 5|x|.$$

Given $\varepsilon > 0$, take $\delta = \min\{1, \varepsilon/5\}$. If $0 < |x| < \delta$, the assumed estimate applies and $|f(x) - 1| \leq 5|x| < \varepsilon$.

3. No. The constant function $g(x) = 1/2$ has limit $1/2$ and satisfies the bounds. In contrast, on $(-1, 0) \cup (0, 1)$ define

$$g(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

This also satisfies the bounds, but its left-hand and right-hand limits are 0 and 1. They disagree, so the two-sided limit does not exist. The two constant bounds have different limits, so the squeeze theorem does not force a conclusion.

4. For nonzero x sufficiently close to zero,

$$\frac{\sin(5x)}{\tan(2x)} = \frac{5}{2} \frac{\sin(5x)}{5x} \frac{2x}{\sin(2x)} \cos(2x).$$

The last three factors tend to $1, 1, 1$, using the standard sine limit, the reciprocal law and cosine continuity. The answer is $\boxed{5/2}$.

5. For $x \neq 0$, the half-angle identity gives

$$\frac{1 - \cos x}{x^2} = \frac{2 \sin^2(x/2)}{x^2} = \frac{1}{2} \left(\frac{\sin(x/2)}{x/2} \right)^2 \longrightarrow \frac{1}{2}.$$

Continuity at zero requires this limit to equal $F(0) = c$, so $\boxed{c = 1/2}$.

6. Since $-1 \leq \sin x \leq 1$, the numerator is at least 1. Given $M > 0$, take $\delta = 1/\sqrt{M}$. If $0 < |x-1| < \delta$, then

$$\frac{2 + \sin x}{(x-1)^2} \geq \frac{1}{(x-1)^2} > M.$$

This proves the infinite limit directly from its definition.

7. Fix $\delta > 0$ and an arbitrary $M > 0$. Choose a positive integer n so large that

$$u = \frac{\pi}{2} + 2\pi n > \max \left\{ \frac{1}{\delta}, \sqrt{\frac{M}{2}} \right\}.$$

For $x = 1/u$, we have $0 < x < \delta$, $\sin(1/x) = 1$, and

$$f(x) = 2u^2 > M.$$

Thus no upper bound works on any deleted interval around zero.

To disprove the limit $+\infty$, fix $M_0 = 1$. For every $\delta > 0$, choose a positive integer n such that $v = 3\pi/2 + 2\pi n > 1/\delta$. At $x = 1/v$,

$$0 < x < \delta, \quad \sin(1/x) = -1, \quad f(x) = 0 \leq M_0.$$

This is the negation of the defining condition for $+\infty$. Arbitrarily large values near zero do not imply that every sufficiently nearby value is large.

8. The domain is $\mathbb{R} \setminus \{-2, 1\}$. On this domain,

$$r(x) = \frac{x+1}{x+2} = 1 - \frac{1}{x+2}.$$

At $x = 1$, the limit is $\boxed{2/3}$. The graph has a hole at $(1, 2/3)$, because the original domain excludes 1. At $x = -2$,

$$\lim_{x \rightarrow -2^-} r(x) = +\infty, \quad \lim_{x \rightarrow -2^+} r(x) = -\infty.$$

Indeed, $1/(x+2)$ tends to $-\infty$ from the left and $+\infty$ from the right. Hence $x = -2$ is a vertical asymptote. At both ends $1/(x+2) \rightarrow 0$, so $r(x) \rightarrow 1$. The only horizontal asymptote is $\boxed{y = 1}$.

9. For $x \neq 0$,

$$\frac{\sqrt{9x^2 + 1}}{x} = \frac{|x|}{x} \sqrt{9 + \frac{1}{x^2}}.$$

The square-root factor tends to 3 at both ends, by root continuity. The sign factor is 1 for positive x and -1 for negative x . The two requested limits are therefore $\boxed{3}$ and $\boxed{-3}$, respectively. The distinction comes from $\sqrt{x^2} = |x|$.

10. For $x > 3$,

$$\left| \frac{2x+1}{x-3} - 2 \right| = \frac{7}{x-3}.$$

Given $\varepsilon > 0$, choose $A = 3 + 7/\varepsilon > 0$. For every $x > A$, the expression is defined and

$$x - 3 > \frac{7}{\varepsilon} \implies \left| \frac{2x+1}{x-3} - 2 \right| < \varepsilon.$$

This is the definition of the limit at $+\infty$.

11. Polynomial division gives

$$f(x) = x + 2 + \frac{2}{x+1}.$$

Thus $\boxed{m = 1, b = 2}$. At both ends,

$$f(x) - (x + 2) = \frac{2}{x+1} \longrightarrow 0,$$

so $\boxed{y = x + 2}$ is the required oblique asymptote.

12. One choice is

$$\boxed{f(x) = \frac{2x}{x-1} = 2 + \frac{2}{x-1}}.$$

The denominator vanishes only at 1, where the numerator is nonzero, so its domain is exactly $\mathbb{R} \setminus \{1\}$. Also,

$$\lim_{x \rightarrow 1^-} f(x) = -\infty, \quad \lim_{x \rightarrow 1^+} f(x) = +\infty,$$

which verifies the vertical asymptote. Since $2/(x-1) \rightarrow 0$ as $x \rightarrow \pm\infty$, the horizontal asymptote is $y = 2$ at both ends. Finally, substitution gives $f(0) = 0$.

13. For nonzero h sufficiently close to zero, rationalization gives

$$\frac{f(4+h) - f(4)}{h} = \frac{\sqrt{4+h} - 2}{h} = \frac{1}{\sqrt{4+h} + 2}.$$

By root continuity and the quotient law, the last expression tends to $1/4$. Thus $\boxed{f'(4) = 1/4}$, and the tangent is

$$\boxed{y = 2 + \frac{1}{4}(x - 4) = \frac{x}{4} + 1.}$$

14. Since $f(3) = 7$, for $h \neq 0$ sufficiently small we have

$$\frac{f(3+4h) - 7}{h} = 4 \frac{f(3+4h) - f(3)}{4h}.$$

Set $t = 4h$. Then $t \neq 0$ and $t \rightarrow 0$ as $h \rightarrow 0$, so the quotient after the factor 4 tends to $f'(3) = -2$. The requested limit is $\boxed{-8}$. This uses the derivative definition and a change of variable in a limit, not a differentiation rule.

15. Let (a, a^2) be the point of tangency. The derivative calculation in class gives slope $2a$, so the tangent is

$$y = a^2 + 2a(x - a) = 2ax - a^2.$$

It passes through $(0, -4)$ precisely when $-a^2 = -4$, or $a = 2$ or $a = -2$. The two lines and their points of tangency are

$$\boxed{y = 4x - 4 \quad \text{at } (2, 4),} \quad \boxed{y = -4x - 4 \quad \text{at } (-2, 4).}$$

Every point of the parabola has the form (a, a^2) and finite derivative $2a$; every possible tangent through the specified point therefore gives one of the two solutions of $a^2 = 4$. There are no others.