

Homework 4: Solutions

1. Continuity at zero means

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in [0, \infty) : 0 \leq x < \delta \implies |f(x) - 2| < \varepsilon.$$

Its negation is

$$\exists \varepsilon_0 > 0 \forall \delta > 0 \exists x \in [0, \infty) : 0 \leq x < \delta \text{ and } |f(x) - 2| \geq \varepsilon_0.$$

Since $f(0) = 2$, any point witnessing the negation is in fact positive.

2. For $x < 2$, the quotient is $-(x + 2)$; for $x > 2$, it is $x + 2$. Therefore

$$\lim_{x \rightarrow 2^-} f(x) = -4, \quad \lim_{x \rightarrow 2^+} f(x) = 4.$$

The one-sided limits differ, so no two-sided limit exists.

3. For $x \geq -5$ with $x \neq 4$, rationalization gives

$$\frac{\sqrt{x+5} - 3}{x - 4} = \frac{1}{\sqrt{x+5} + 3}.$$

The expression on the right is continuous on $[-5, \infty)$, with right-hand continuity at -5 , because its denominator is at least 3. Its value at 4 is $1/6$. Thus $\boxed{k = 1/6}$ makes F_k equal to that continuous function on its entire domain. Any other value disagrees with the limit at 4.

4. Continuity at zero requires $b = 0$, since the left-hand limit and value are zero and the right-hand limit is b . At 2, the left-hand limit is $2m + b$, whereas the right-hand limit and value are 3. Hence $2m + b = 3$, giving

$$\boxed{m = \frac{3}{2}, \quad b = 0.}$$

Conversely, these values match both joins. Each branch is continuous on its own interval, so the resulting function is continuous on $\mathbb{R}$.

5. The numerator requires $-2 \leq x \leq 2$, and the denominator root requires $x \geq -2$. Its denominator vanishes exactly when $\sqrt{x+2} = 1$, or $x = -1$. Thus the domain is

$$\boxed{[-2, -1) \cup (-1, 2].}$$

Polynomial and root continuity, composition and the quotient theorem show continuity at every point of this domain. At -2 the approach is from the right, and at 2 it is from the left; the denominator values there are -1 and 1 , respectively. The point -1 is excluded, so no continuity assertion there is required.

6. Here $f(1) = 0$. Given $\varepsilon > 0$, choose $\delta = \varepsilon^4$. If $x \leq 1$ and $|x - 1| < \delta$, then

$$|f(x) - f(1)| = (1 - x)^{1/4} = |x - 1|^{1/4} < \delta^{1/4} = \varepsilon.$$

This proves continuity within the domain, including $x = 1$.

7. Let $H = g - f$. The arithmetic continuity theorem gives continuity of H at a , and $H(a) = g(a) - f(a) > 0$. Apply the definition with $\varepsilon = H(a)/2$ to obtain $r > 0$ such that

$$x \in I, \quad |x - a| < r \quad \implies \quad |H(x) - H(a)| < H(a)/2.$$

Then $H(x) > H(a)/2 > 0$, which is $g(x) > f(x)$.

8. The function $(f - g)^2$ is continuous and nonnegative on I . The square-root function is continuous on $[0, \infty)$, including at zero. Composition therefore makes $\sqrt{(f - g)^2}$ continuous. The given identity now expresses H as a sum and scalar multiple of continuous functions, so H is continuous on I . This also covers points where $f(x) = g(x)$.

9. One choice is

$$g(t) = \begin{cases} 1, & t < 0, \\ 0, & t \geq 0. \end{cases}$$

Its left-hand limit at zero is 1, while its value and right-hand limit are zero. Thus g is discontinuous there. But $x^2 \geq 0$ for every real x , so $g(f(x)) = 0$ everywhere. The composition is a constant function and hence is continuous at zero.

10. The value of the composition at 1 is $g(f(1)) = -1$. For $\varepsilon > 0$, choose

$$\delta = \min \left\{ \frac{1}{3}, \frac{\varepsilon}{6} \right\} > 0.$$

If $|x - 1| < \delta$, then $|x - 1| < 1$, and

$$|f(x) - 2| \leq 3|x - 1| < 3\delta \leq 1.$$

Consequently the second assumed estimate applies with $y = f(x)$:

$$|g(f(x)) - g(f(1))| = |g(f(x)) + 1| \leq 2|f(x) - 2| \leq 6|x - 1| < 6\delta \leq \varepsilon.$$

The bound $\delta \leq 1/3$ ensures that the outer estimate is applicable.

11. Let $P(x) = x^3 - 4x + 1$. It is continuous on $\mathbb{R}$, and

$$P(-3) = -14, \quad P(-2) = 1, \quad P(0) = 1, \quad P(1) = -2, \quad P(2) = 1.$$

The signs differ at the endpoints of each of

$$\boxed{(-3, -2), \quad (0, 1), \quad (1, 2).}$$

The Intermediate Value Theorem gives a root in each interval. These open intervals are disjoint, so the three roots are distinct.

12. The denominator never vanishes on D , so the rational function is continuous at each point of D , using the appropriate one-sided interpretation at the four interval endpoints. Its endpoint values are $f(-2) = -1/2$ and $f(2) = 1/2$, but $1/x \neq 0$ throughout D . The theorem requires a function continuous on the whole interval $[-2, 2]$. Here f is not defined at the points in the gap $(-1, 1)$. Continuity on two separated intervals does not meet that hypothesis.

13. Set $H(x) = f(x) - x$. It is continuous on $[0, 1]$, and the restriction on the values of f gives

$$H(0) = f(0) \geq 0, \quad H(1) = f(1) - 1 \leq 0.$$

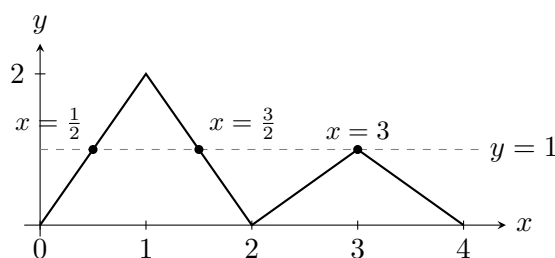
If either endpoint value of H is zero, that endpoint supplies the required c . Otherwise $H(0) > 0 > H(1)$, and the Intermediate Value Theorem supplies a $c \in (0, 1)$ with $H(c) = 0$. In all cases, $f(c) = c$.

14. The equation $f(x)^2 = 1$ forces each value to be 1 or -1 . If both values occurred, there would be two points $u < v$ in I with opposite function values. Since I is an interval, it contains $[u, v]$. Continuity and the Intermediate Value Theorem would give a $c \in (u, v)$ with $f(c) = 0$, contradicting $f(c)^2 = 1$. Thus both signs cannot occur. Since I is nonempty, all values are 1, or all are -1 .

15. Join the successive points

$$(0, 0), \quad (1, 2), \quad (2, 0), \quad (3, 1), \quad (4, 0)$$

by line segments. They meet at their common endpoints, giving a continuous function on $[0, 4]$ with the required endpoint values.



The first two segments have equations $y = 2x$ and $y = 4 - 2x$, so they meet $y = 1$ exactly at $x = 1/2$ and $x = 3/2$. The last two have equations $y = x - 2$ and $y = 4 - x$; each reaches $y = 1$ only at their common endpoint $x = 3$. These are exactly three distinct solutions.