

Homework 3: Solutions

1. The numerator tends to $2^2 + 3(-1) = 1$, and the denominator tends to $2(2) - (-1) = 5$. The nonzero-limit lemma ensures that $|2f(x) - g(x)| > 5/2$ sufficiently near a , with a excluded. The quotient law therefore gives $\boxed{1/5}$.

2. The original denominator tends to zero, so the quotient law does not apply in that form. For $x \neq 1$,

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1.$$

The polynomial on the right has limit 3. The two functions agree at every point used in the limit definition, so the original limit is $\boxed{3}$.

3. If the quotient tends to 5, multiplying by $x - 2$ shows that $x^2 + cx + d$ tends to 0. Its polynomial limit is $4 + 2c + d$, so $d = -4 - 2c$. With this necessary condition,

$$x^2 + cx + d = (x - 2)(x + c + 2).$$

The quotient equals $x + c + 2$ for $x \neq 2$, hence its limit is $c + 4$. Requiring $c + 4 = 5$ gives $c = 1$ and $d = -6$. Conversely, these values give the quotient $x + 3$ away from 2, whose limit is 5. Thus $\boxed{c = 1, d = -6}$ is the unique pair.

4. Write $A = 2f + g$ and $B = f - g$. The pointwise identities

$$f = \frac{A + B}{3}, \quad g = \frac{A - 2B}{3}$$

express both functions using expressions whose limits are given. The sum, difference and scalar-multiple laws prove that the limits exist, and yield

$$\lim_{x \rightarrow a} f(x) = \frac{7 + 2}{3} = 3, \quad \lim_{x \rightarrow a} g(x) = \frac{7 - 4}{3} = 1.$$

5. Given $\varepsilon > 0$, set $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x - 1| < \delta$, then $0 < x < 2$, so $x + 1 > 1$. In particular, the function is defined, and

$$\left| \frac{1}{x+1} - \frac{1}{2} \right| = \frac{|x-1|}{2|x+1|} \leq \frac{|x-1|}{2} < \frac{\delta}{2} \leq \varepsilon.$$

This proves the limit directly.

6. Apply the limit definition with $\varepsilon = 1$ to obtain $r > 0$ such that $|f(x) + 3| < 1$ whenever $0 < |x - a| < r$. This gives $-4 < f(x) < -2$, hence $2 < |f(x)| < 4$. Taking reciprocals of these positive numbers gives $1/4 < 1/|f(x)| < 1/2$.
7. Take $f(x) = x$. Its limit at zero is zero: given $\varepsilon > 0$, the choice $\delta = \varepsilon$ proves the required inequality. For any $B > 0$, take $x = B + 1$; then $|f(x)| = B + 1 > B$. A limit gives boundedness only near the point.
8. For $0 < |x| < 1$, we have $|f(x)| \leq |f(x) - 2| + 2 < 5$. Since $f(x)g(x) + 2 = f(x)(g(x) + 1) - (f(x) - 2)$,

$$\begin{aligned} |f(x)g(x) + 2| &\leq |f(x)| |g(x) + 1| + |f(x) - 2| \\ &\leq 5 \cdot 2|x| + 3|x| = 13|x|. \end{aligned}$$

Given $\varepsilon > 0$, choose $\delta = \min\{1, \varepsilon/13\}$. For $0 < |x| < \delta$, the estimate gives $|f(x)g(x) + 2| \leq 13|x| < 13\delta \leq \varepsilon$.

9. Let $\varepsilon > 0$. The limit of g gives $\delta_1 > 0$ such that $|g(x)| < \varepsilon/4$ when $0 < |x - a| < \delta_1$. Set $\delta = \min\{r, \delta_1\}$. Then

$$|f(x)g(x)| \leq 4|g(x)| < \varepsilon$$

for $0 < |x - a| < \delta$. This uses a bound on f , not a limit of f . For the application, take $f(x) = \sin(1/x)$ and $g(x) = x$. Since $|f(x)| \leq 1 \leq 4$ and $g(x) \rightarrow 0$, the limit is $\boxed{0}$. Directly, $|x \sin(1/x)| \leq |x|$, so $\delta = \varepsilon$ also proves it.

10. Given $\varepsilon > 0$, choose $\delta > 0$ from the limit of f so that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$. For all these points,

$$||f(x)| - |L|| \leq |f(x) - L| < \varepsilon.$$

This is the required limit definition for $|f|$.

11. Choose $P(x) = Lx$ and $Q(x) = x$. Both have limit zero at zero, and $Q(x) \neq 0$ for $x \neq 0$. At every such point, $P(x)/Q(x) = L$, so its limit is L . The construction also works for $L = 0$, when P is the zero polynomial.
12. Fix any $L \in \mathbb{R}$ and set $\varepsilon_0 = 1/2$. For a given $\delta > 0$, put $h = \min\{\delta/2, 1/4\}$, so $0 < h < \delta$. If $L \leq 2$, choose $x = h$. Then

$$f(x) - L = 3 - h - L \geq 1 - h \geq 3/4 > \varepsilon_0.$$

If $L > 2$, choose $x = -h$. In this case $f(x) = 1 - h$, so

$$|f(x) - L| = L - 1 + h > 1 > \varepsilon_0.$$

In both cases $0 < |x| < \delta$ and the error is at least ε_0 . Every finite candidate therefore fails.

- 13.** Fix $L \in \mathbb{R}$ and take $\varepsilon_0 = 1$. Given $\delta > 0$, choose a positive integer n such that $2\pi n > 1/\delta^2$. Define

$$u = \frac{1}{\sqrt{2\pi n}}, \quad v = \frac{1}{\sqrt{(2n+1)\pi}}.$$

Both points lie in $(0, \delta)$. The function equals 1 at u and -1 at v . If $L \leq 0$, use $x = u$, whose error is $1 - L \geq 1$. If $L > 0$, use $x = v$, whose error is $1 + L > 1$. This disproves every finite candidate for every proposed δ .

- 14.** Take $f(x) = |x|/x$ and $g(x) = -|x|/x$ for $x \neq 0$. The non-existence of the limit of f was proved in Lecture 4. If g had a finite limit, the scalar-multiple law would give a finite limit for $-g = f$, a contradiction. Nevertheless,

$$f(x) + g(x) = 0, \quad f(x)g(x) = -1$$

for every $x \neq 0$, so the two limits are 0 and -1 . The limit laws do not assert their converses.

- 15.** Suppose instead that $f(x) \rightarrow L \in \mathbb{R}$. The product law would give

$$\lim_{x \rightarrow a} \left(f(x) \frac{1}{f(x)} \right) = L \cdot 0 = 0.$$

But this product equals 1 on a punctured neighborhood of a , so its limit is 1. This contradicts uniqueness of limits. Thus f has no finite limit at a .