

Homework 2: Solutions

1. The definition is: for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every x in the domain,

$$0 < |x + 1| < \delta \implies |f(x) - 2| < \varepsilon.$$

Its negation is: there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists a domain point x with

$$0 < |x + 1| < \delta \quad \text{and} \quad |f(x) - 2| \geq \varepsilon_0.$$

The positive tolerance ε_0 must be fixed first. The point x may depend on δ .

2. Given $\varepsilon > 0$, choose $\delta = \varepsilon/(|m| + 1) > 0$. For $0 < |x - a| < \delta$,

$$|(mx + b) - (ma + b)| = |m||x - a| \leq (|m| + 1)|x - a| < \varepsilon.$$

This choice works for every m , including zero. Alternatively, use $\varepsilon/|m|$ when $m \neq 0$ and any positive δ when $m = 0$.

3. If $|x + 1| < 1$, then

$$|x - 1| = |(x + 1) - 2| \leq |x + 1| + 2 < 3.$$

Given $\varepsilon > 0$, choose $\delta = \min\{1, \varepsilon/3\}$. For $0 < |x + 1| < \delta$,

$$|x^2 - 1| = |x + 1||x - 1| < 3|x + 1| < 3\delta \leq \varepsilon.$$

4. Choose $\delta = \sqrt{\varepsilon}$. If $0 < |x| < \delta$, then

$$|x^2| = |x|^2 < \delta^2 = \varepsilon.$$

No additional fixed restriction on δ is needed.

5. The limit is 4, whereas $f(1) = 100$. Given $\varepsilon > 0$, take $\delta = \min\{1/2, \varepsilon/3\}$. If $0 < |x - 1| < \delta$, the first branch applies, and

$$|f(x) - 4| = |3x + 1 - 4| = 3|x - 1| < 3\delta \leq \varepsilon.$$

The values outside this neighborhood, and at 1, do not enter the estimate.

6. One choice is $f(x) = 2 + |x|$ for $x \neq 0$. It satisfies $f(x) > 2$ at every point of its domain. Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $0 < |x| < \delta$ implies $|f(x) - 2| = |x| < \varepsilon$, so its limit is 2.
7. Uniqueness of limits gives $2c - 1 = 5 - c$, hence $c = 2$. Both expressions then equal 3, so the common limit is 3.

8. Choose

$$\delta = \min\{1, \varepsilon/8, \varepsilon^2/4\}.$$

For $0 < |x - 1| < \delta$, both assumed bounds hold and

$$\begin{aligned} |f(x) + g(x) + 1| &\leq |f(x) - 2| + |g(x) + 3| \\ &\leq 4|x - 1| + \sqrt{|x - 1|} < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

9. Repeated application of the sum law gives

$$(a) \ 2 + (-1) + 4 + 5 = 10, \quad (b) \ 2 + 2 + (-1) = 3.$$

The constant function 5 has limit 5. In (b), the two occurrences of f are simply two summands; no scalar-multiple law is needed.

10. Take $\varepsilon_0 = 1/2$. For any $\delta > 0$, choose $x = \min\{\delta/2, 1/2\}$. Then $0 < x < \delta$, and $x^2 \leq 1/4$, so $|x^2 - 1| \geq 3/4 \geq \varepsilon_0$. This disproves the proposed limit 1. It does not assert that no limit exists; Exercise 4 proves the actual limit is 0.