

Homework 1: Solutions

1. (a) $A = (-\sqrt{3}, \sqrt{3})$, so $\sup A = \sqrt{3}$, which is not in A . It is an upper bound, and every smaller number is exceeded by some member of this interval.
- (b) $\sup B = 5/2$, which is not in B . Every rational in B is smaller than $5/2$; between any proposed upper bound smaller than $5/2$ and $5/2$ lies a rational in B .
- (c) $\sup C = 4$, which belongs to C and is its largest member.

2.

$$(a) (3/2)^{-3} = \frac{8}{27}, \quad (b) (2^5)^{2/5} = 4, \quad (c) 25^{(1/3)(3/2)} = 5.$$

3.

$$(a) a^{(\sqrt{5}-1)(\sqrt{5}+1)-2} = a^{4-2} = a^2,$$

$$(b) \frac{x^3 y^{-1}}{x y^{-2}} = x^2 y.$$

The assumed positivity permits all the power laws used.

4. Let $u = \sqrt[n]{a}$ and $v = \sqrt[n]{b}$. By definition, $u, v > 0$, $u^n = a$, and $v^n = b$. Thus $uv > 0$, and integer power laws give

$$(uv)^n = u^n v^n = ab.$$

Since ab has a unique positive n th root, $uv = \sqrt[n]{ab}$, as required.

5. Since $1/4 = (1/2)^2$ and the exponential function with base $1/2$ is decreasing, the inequality is equivalent to

$$x^2 - x \leq 2, \quad (x-2)(x+1) \leq 0.$$

The product is nonpositive exactly when $-1 \leq x \leq 2$. The solution is $[-1, 2]$.

6. Set $t = 2^x > 0$. Then

$$t^2 - 5t + 4 = (t - 1)(t - 4) = 0.$$

Thus $2^x = 1$ or $2^x = 4$, giving $x = 0$ or $x = 2$. Both satisfy the original equation.

7. The original domains are $D_f = [-2, \infty)$ and $D_g = (-\infty, 3]$.

(a) $(f + g)(x) = \sqrt{x + 2} + \sqrt{3 - x}$, domain $[-2, 3]$.

(b) $(fg)(x) = \sqrt{x + 2}\sqrt{3 - x}$, domain $[-2, 3]$.

(c) $(-2f)(x) = -2\sqrt{x + 2}$, domain $[-2, \infty)$.

(d) $(f/g)(x) = \sqrt{x + 2}/\sqrt{3 - x}$, domain $[-2, 3)$. The endpoint 3 is excluded because the denominator is zero there.

8.

$$(f \circ g)(x) = \sqrt{\frac{1}{x - 2}}, \quad D_{f \circ g} = (2, \infty).$$

The reciprocal must be defined and nonnegative, which here requires $x > 2$. In the other order,

$$(g \circ f)(x) = \frac{1}{\sqrt{x - 2}}, \quad D_{g \circ f} = [0, 4) \cup (4, \infty).$$

The conditions are $x \geq 0$ and $\sqrt{x} \neq 2$. The two functions have different domains, so they are not equal.

9. No. Although $F(x) = G(x)$ for every $x \neq 3$, their domains differ: $G(3) = 6$, while $F(3)$ is undefined. Restrict G to $\mathbb{R} \setminus \{3\}$ to obtain the same function as F .

10. Since $x + 2 \leq 0$, solving $y + 3 = (x + 2)^2$ gives

$$f^{-1}(y) = -2 - \sqrt{y + 3}, \quad y \geq -3.$$

The inverse domain is $[-3, \infty)$ and its range is $(-\infty, -2]$. For $x \leq -2$,

$$f^{-1}(f(x)) = -2 - \sqrt{(x + 2)^2} = -2 - |x + 2| = x.$$

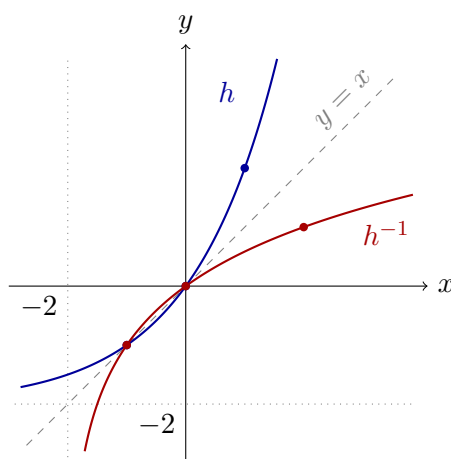
For $y \geq -3$,

$$f(f^{-1}(y)) = (-\sqrt{y + 3})^2 - 3 = y.$$

11. (a) The equation $(x+1)/(x-1) = 1$ would imply $x+1 = x-1$, which is impossible. Thus the composition is defined for every $x \neq 1$, and

$$f(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{2x/(x-1)}{2/(x-1)} = x.$$

- (b) Every $y \neq 1$ is attained at input $f(y)$, since $f(f(y)) = y$. Hence the range is $\mathbb{R} \setminus \{1\}$. If $f(u) = f(v)$, applying f gives $u = v$, so f is one-to-one. Therefore $f^{-1} = f$. It is not the identity: for example, $f(2) = 3$.
12. (a) The range of h is $(-2, \infty)$. Solving $y+2 = 2^{x+1}$ gives $h^{-1}(y) = \log_2(y+2) - 1$, with domain $(-2, \infty)$ and range $\mathbb{R}$.
- (b) Points on h are $(-1, -1)$, $(0, 0)$, and $(1, 2)$. Points on h^{-1} are $(-1, -1)$, $(0, 0)$, and $(2, 1)$. Both graphs have their only axis intercept at $(0, 0)$. The asymptotes are $y = -2$ for h and $x = -2$ for h^{-1} .



13. The values are

$$(a) \frac{3}{2}, \quad (b) -\frac{3}{2}, \quad (c) \frac{7}{2}, \quad (d) 1.$$

Indeed, $9^{3/2} = 27$, $(1/4)^{-3/2} = 8$, $e^3 \sqrt{e} = e^{7/2}$, and $2 \ln \sqrt{7} = \ln 7$.

14. The numerator is positive except at $x = 1$, and the denominator must be positive. Thus

$$D_L = (-2, 1) \cup (1, \infty), \quad L(x) = 2 \ln |x - 1| - \ln(x + 2).$$

The expression $2 \ln(x - 1) - \ln(x + 2)$ excludes the valid interval $(-2, 1)$, because $x - 1 < 0$ there.

15. (a) The original domain is $x > 1$. The equation becomes

$$(x + 2)(x - 1) = 4, \quad x^2 + x - 6 = (x + 3)(x - 2) = 0.$$

Only $x = 2$ lies in the domain; it gives $\ln 4 + \ln 1 = \ln 4$.

- (b) The original domain is $x > 0$. The equation becomes

$$x^2 = 3x + 4, \quad (x - 4)(x + 1) = 0.$$

Only $x = 4$ is allowed; $2 \ln 4 = \ln 16 = \ln(3 \cdot 4 + 4)$.

16. Require $x > 1$. Since $-2 = \log_{1/2} 4$ and the logarithm with base $1/2$ is decreasing,

$$\log_{1/2}(x - 1) \geq \log_{1/2} 4 \iff 0 < x - 1 \leq 4.$$

The solution is $(1, 5]$. A decreasing function reverses the order of its inputs.

17. (a) By change of base,

$$\log_a b = \frac{\ln b}{\ln a}, \quad \log_b a = \frac{\ln a}{\ln b}.$$

Both logarithms in the fractions are nonzero because $a, b \neq 1$. The two values are reciprocals.

- (b) Similarly, $\ln v \neq 0$, so

$$\frac{\log_a u}{\log_a v} = \frac{\ln u / \ln a}{\ln v / \ln a} = \frac{\ln u}{\ln v} = \log_v u.$$

18. The values are

$$(a) -\frac{\pi}{3}, \quad (b) \frac{2\pi}{3}, \quad (c) \frac{\pi}{4}, \quad (d) \frac{2\pi}{3}.$$

These angles have the required sine, cosine, tangent, or secant and lie in the specified inverse ranges. In particular, $\sec(2\pi/3) = -2$.

19. (a) $\sin(7\pi/6) = -1/2$, so the selected angle is $-\pi/6 \in [-\pi/2, \pi/2]$.

(b) $\cos(7\pi/4) = \sqrt{2}/2$, so the selected angle is $\pi/4 \in [0, \pi]$.

(c) $\tan(5\pi/6) = -1/\sqrt{3}$, so the selected angle is $-\pi/6 \in (-\pi/2, \pi/2)$.

(d) $\sec(5\pi/4) = -\sqrt{2}$, so the selected angle is $3\pi/4 \in [0, \pi] \setminus \{\pi/2\}$.

20. (a) Set $\theta = \arctan x$. Its range gives $\cos \theta > 0$. Since $1 + \tan^2 \theta = 1/\cos^2 \theta$,

$$\cos \theta = \frac{1}{\sqrt{1+x^2}}, \quad \sin(\arctan x) = \tan \theta \cos \theta = \frac{x}{\sqrt{1+x^2}}.$$

The domain is $\mathbb{R}$. The formula includes negative x ; the sign of the sine is carried by the numerator.

(b) Set $\theta = \arccos x$, so $\theta \in [0, \pi]$ and $\sin \theta \geq 0$. Hence $\sin \theta = \sqrt{1-x^2}$ and

$$\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}.$$

The domain is $[-1, 0) \cup (0, 1]$: $\arccos x$ requires $-1 \leq x \leq 1$, and tangent is undefined at $\theta = \pi/2$, which corresponds to $x = 0$. Both endpoints $x = \pm 1$ are allowed.